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RANGE KERNEL AND ELEMENTARY OPERATORS

SALAH MECHERI

ABSTRACT : Let $B(H)$ denote the algebra of all bounded linear operators on a separable infinite dimensional complex Hilbert space H into itself. Let $A = (A_1, A_2, \ldots, A_n)$ and $B = (B_1, B_2, \ldots, B_n)$ be n-tuples in $B(H)$, we define the elementary operator $\Delta_{A,B} : B(H) \to B(H)$ by $\Delta_{A,B}(X) = \sum_{i=1}^{n} A_i X B_i - X$. In this paper we initiate the study of some properties of the range of such operators. Other related results are also given.

2000 Mathematics Subject Classification : Primary 47B47, 47A30, 47B20; Secondary 47B10.

Key words : Generalized Derivation, Elementary Operators, Trace Class operators, Finite rank Operators.

1. INTRODUCTION

Let $B(H)$ be the algebra of all bounded linear operators acting on a complex separable Hilbert space H. The generalized derivation operator $\delta_{A,B}$ associated with (A, B), defined on $B(H)$ by

$$\delta_{A,B}(X) = AX - XB$$

was initially systematically studied by M. Rosenblum [14]. The properties of such operators have been much studied (see for example [1], [2], [4], [12], [13] and [14]).

If $A = B$, $\delta_{A,A} = \delta_A : B(H) \to B(H)$ is called the inner derivation defined by

$$\delta_A(X) = AX - XA.$$

The theory of derivations has been extensively dealt with in the literature (see, [7], [8], [10], [11], [16], [17], [18] and [19]).

In this paper we initiate the study of some properties of (the related derivation)

$$\Delta_{A,B} : B(H) \to B(H)$$

defined by

$$\Delta_{A,B}(X) = AXB - X.$$

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Noting $\Delta_{A,A} = \Delta_A = AXA - X$. In [20, p.276], J. P. Williams showed that if, $A \in B(H)$, then

$$R(\delta_A)^{\circ} \simeq R(\delta_A)^{\circ} \cap K^{\circ}(H) \oplus \ker(\delta_A) \cap C_1,$$

where $R(\delta_A)$, $K(H)$, $\ker(\delta_A)$ and C_1 denote, respectively, the range of δ_A, the ideal of compact operators, the kernel of δ_A and the trace class operators. This result extends to $\delta_{A,B}$ has been carried out by S. Bouali [5]. Let $A = (A_1, A_2, \ldots, A_n)$ and $B = (B_1, B_2, \ldots, B_n)$ be n-tuples in $B(H)$, we define the elementary operator $\Delta_{A,B} : B(H) \to B(H)$ by $\Delta_{A,B}(X) = \sum_{i=1}^{n} A_i X B_i - X$. In order to generalize all the above results we prove its in the case where $\delta_{A,B}$ is replaced by $\Delta_{A,B}$. Which leads us to show that the equivalence of the following statements :

(a) $K(H) \subset \overline{R(\Delta_{A,B})}$;

(b) $\overline{R(\Delta_{A,B})}^{w^*} = B(H)$;

(c) $\ker \Delta_{B,A} \cap C_1 = \{0\}$.

Where $\overline{R(\Delta_{A,B})}$, $\overline{R(\Delta_{A,B})}^{w^*}$ denote, respectively the closure of $R(\Delta_{A,B})$ in the norm topology and the closure of $R(\Delta_{A,B})$ in the weak*-topology. By using this result we deduce that $R(\Delta_{A,B})$ is weakly dense in $B(H)$, if and only if,

$$\ker \Delta_{B,A} \cap F(H) = \{0\}.$$

where $F(H)$ is the class of all finite rank operators, and we prove that if $A, B \in B(H)$, then

$$\overline{R(\Delta_{A,B})} \cap K(H) = \overline{R(\Delta_{A,B})}^{w^*} \cap K(H).$$

In the second part of this paper we characterize all finite rank operators in

$$\overline{R(\Delta_{A,B})}^{w} \cap \ker \Delta_{A^*,B^*}$$

and in

$$\overline{R(\Delta_{A,B})}^{w^*} \cap \ker \Delta_{A^*,B^*}$$

and we show that $R(\Delta_{A,B})$ is w-dense in $B(H)$, if and only if,

$$F(H) \cap \ker \Delta_{B,A} = \{0\}.$$

We also prove that $R(\Delta_{A,B})$ is w*-dense in $B(H)$, if and only if,

$$C_1 \cap \ker \Delta_{B,A} = \{0\},$$

where $F(H)$, $\overline{R(\Delta_{A,B})}^{w}$ and $\overline{R(\Delta_{A,B})}^{w^*}$ denote, respectively, finite rank operators, the trace class operators, the closure of the range of $\Delta_{A,B}$ in the weak operator topology and the closure of the range of $\Delta_{A,B}$ in the ultra-weak operator topology.

2. MAIN RESULTS

Note that the weak continuous linear form (resp. the ultra-weak continuous linear form) on $B(H)$ are defined by Φ_T, where $T \in F(H)$ (resp. $T \in C_1$) with

$$\Phi_T(X) = tr(XT) = tr(TX)$$

for all $X \in B(H)$ (see [6], p. 23).

Let S be a sub-vector space of $B(H)$. Noting by

$S^0 = \{f \in B(H)' : f(x) = o, \text{ if } x \in S\}$.

Lemma 2.1. *Let S_1, S_2 be two sub-vectorspaces of $B(H)$.*

Then $S_1^0 \subset S_2^0$ if and only if $S_2 \subset \bar{S}_2$.

Theorem 2.1. *Let $A = (A_1, A_2, ..., A_n)$ and $B = (B_1, B_2, ..., B_n)$ be n-tuples of operators in $B(H)$, then*

$$R(\Delta_{A,B})^\circ \simeq R(\Delta_{A,B})^\circ \cap K^\circ(H) \oplus \ker(\Delta_{B,A}) \cap C_1.$$

Proof. If $\Phi \in R(\Delta_{A,B})^\circ$, i.e., $\Phi(\sum_{i=1}^{n} A_i X B_i - X) = 0$, we have $\Phi = \Phi_0 + \Phi_T$ with $\Phi_0 \in K^\circ(H)$ and $T \in C_1$ (see [19]). Choose X to be the rank one operator $x \otimes y$ for some arbitrary elements x and y in H, then

$$\Phi(\sum_{i=1}^{n} A_i X B_i - X) = \Phi_T(\sum_{i=1}^{n} A_i X B_i - X) = tr[T(\sum_{i=1}^{n} A_i X B_i - X)] =$$

$$tr[(\sum_{i=1}^{n} B_i T B_i - T)X] = 0$$

implies

$$<\Delta_{B,A}(T)x, y> = 0$$

equivalently

$$T \in \ker \Delta_{B,A}.$$

Since

$$\Phi_T\left[\sum_{i=1}^{n} A_i(x \otimes y)B_i - (x \otimes y)\right] = 0$$

for all x, $y \in H$, it results that

$$\Phi_T(\sum_{i=1}^{n} A_i X B_i - X) = 0$$

for all finite rank X. It is well known that the set of finite rank operators is w^*–dense in $B(H)$, then it follows that

$$\Phi_T(\sum_{i=1}^{n} A_i X B_i - X) = 0$$

for all $X \in B(H)$, i.e., $\Phi_T \in R(\Delta_{A,B})^\circ$. Hence

$$\Phi_0 = \Phi - \Phi_T \in R(\Delta_{A,B})^\circ.$$

Conversely assume that $T \in \ker \Delta_{B,A}$, then the above calculation shows that

$$\Phi_T \in R(\Delta_{A,B})^\circ.$$

Theorem 2.2. *Let A, $B \in B(H)$, then the following statements are equivalent*

(a) $K(H) \subset \overline{R(\Delta_{A,B})}$;

(b) $\overline{R(\Delta_{A,B})}^{w^*} = B(H)$;

(c) $\ker \Delta_{B,A} \cap C_1 = \{0\}$.

Proof. The negation of (b) and (c) is equivalent to the fact that, there exists a non-zero linear form Φ_T, w^*-continuous such that

$$\Phi_T \in R(\Delta_{A,B})^\circ,$$

or equivalently (by Theorem 2.2), to

$$R(\Delta_{A,B})^\circ \subseteq K^0(H)$$

and Lemma 2.1 would imply that

$$K(H) \subseteq \overline{R(\Delta_{A,B})}.$$

Corollary 2.1. Let $A,\ B \in B(H)$. *Then*

$$\overline{R(\Delta_{A,B})}^{w} = B(H),$$

if and only if,

$$\ker \Delta_{B,A} \cap F(H) = \{0\}.$$

Proof. It is well known that $R(\Delta_{A,B})$ non w–dense equivalent to the existence of a non zero w–continuous linear form Φ_T, such that $\Phi_T \in R(\Delta_{A,B})^{\circ}$. Which is equivalent by Theorem 2.2 to $\ker \Delta_{B,A} \cap F(H) = \{0\}$.

Corollary 2.2 *Let* $A,\ B \in B(H)$. *Then*

$$\overline{R(\Delta_{A,B})} \cap K(H) = \overline{R(\Delta_{A,B})}^{w^*} \cap K(H).$$

Proof. Let $K \in \overline{R(\Delta_{A,B})}^{w^*} \cap K(H)$. If $\Phi \in R(\Delta_{A,B})^{\circ}$, it results from Theorem 2.2 that $\Phi = \Phi_0 + \Phi_T$ with

$$\Phi_0 \in K(H)^{\circ} \cap R(\Delta_{A,B})^{\circ}$$

and

$T \in C_1 \cap \ker \Delta_{B,A}$.

Since

$\Phi_T \in R(\Delta_{A,B})^{\circ} \cap B(H)'^{w*}$,

$\Phi(K) = \Phi_0(K) + \Phi_T(K) = 0$,

i.e.,

$K \in \{\ker \Phi : \Phi \in R(\Delta_{A,B})^{\circ}\} = \overline{R(\Delta_{A,B})}$.

3. FINITE RANK OPERATORS AND ELEMENTARY OPERATORS

Let C_1 be the ideal of trace class operator, that is, all compact operators $A \in B(H)$ for which the eigenvalues of $(TT^*)^{\frac{1}{2}}$ (counted according to multiplicity) are summable. The ideal C_1 admits a complex valued function $tr(T)$ which has the characteristic properties of the trace for matrices. The trace function is defined by $tr(T) = \sum_n (Te_n, e_n)$, where (e_n) is any complete orthonormal sequence in H. As Banach spaces, C_1 may be identified with the conjugate space of the ideal K of compact operators by means of the linear isometry $T \to f_T$, where $f_T(X)$

= $tr(XT)$. Moreover, $B(H)$ is the conjugate space of C_1. The ultraweak continuous linear functionals on $B(H)$ are those of the form f_T for some $T \in C_1$. Furthermore the weak continuous linear functionals on $B(H)$ are those of the form f_T where T is of finite rank (see J. B. Conway [6]).

Theorem 3.1. *Let A, $B \in B(H)$. Then,*

1) every finite rank operator in $\overline{R(\Delta_{A,B})}^{w} \cap \ker \Delta_{A^*,B^*}$ vanishes.

2) every trace class operator in $\overline{R(\Delta_{A,B})}^{w^*} \cap \ker \Delta_{A^*,B^*}$ vanishes.

Proof. Let $T \in \overline{R(\Delta_{A,B})}^{w} \cap \ker \Delta_{A^*,B^*}$, then

$T^* \in \ker \Delta_{A,B} \cap F(H)$; $BT^*A = T^*$.

Since $T \in \overline{R(\Delta_{A,B})}^{w}$, there exists a generalized sequence $\{X_\alpha\}$ such that

$$\sum_{i=1}^{n} A_i X_\alpha B_i - X_\alpha \xrightarrow{w} T$$

hence,

$$\Phi_{T^*}\left(\sum_{i=1}^{n} A_i X_\alpha B_i - X_\alpha\right) \xrightarrow{w} \Phi_{T^*}(T)$$

and we have

$$\Phi_{T^*}\left(\sum_{i=1}^{n} A_i X_\alpha - X_\alpha\right) = tr\left(T^*\left(\sum_{i=1}^{n} A_i X_\alpha B_i - X_\alpha\right)\right) = tr\left[\left(\sum_{=1}^{n} BT^*A - T^*\right)X_\alpha\right] = 0.$$

So,

$$0 = \Phi_{T^*}\left(\sum_{i=1}^{n} A_i X_\alpha B_i - X_\alpha\right) \xrightarrow{w} \Phi_{T^*}(T).$$

Then, $\Phi_{T^*}(T) = tr(TT^*) = 0$, that is, $TT^* = 0$ and thus $T = 0$.

2) It suffices to replace $F(H)$ by C_1 in the above proof.

Remark 3.1. *J. P. Williams [19] has shown that if $A \in B(H)$, then every finite rank operator in $\overline{R(\delta_A)}^w \cap \{A^*\}'$ vanishes. In the general case where δ_A is replaced by $\delta_{A,B}$ this result is carried out by S. Mecheri [13].*

Theorem 3.2. *Let A, $B \in B(H)$. Then,*

1) $\overline{R(\Delta_{A,B})}^w = B(H) \Leftrightarrow \ker \Delta_{B,A} \cap F(H) = 0.$

2) $\overline{R(\Delta_{A,B})}^{w^*} = B(H) \Leftrightarrow \ker \Delta_{B,A} \cap C_1 = 0.$

Proof. 1) Suppose that $\overline{R(\Delta_{A,B})}^w = L(H)$ and $T \in \ker \Delta_{B,A} \cap F(H)$, then

$$T^* \in \overline{R(\Delta_{A,B})}^w \cap \ker \delta_{A^*,B^*},$$

hence $T = 0$ by Theorem 3.1.

Conversely assume that, there exists $T \in B(H) \setminus \overline{R(\Delta_{A,B})}^w$, then there exists $S \in F(H)$ such that $tr(ST) \neq tr(SX) = 0$, for every $X \in R(\Delta_{A,B})$. So $T \in \ker \Delta_{B,A} \cap F(H)$ and $S \neq 0$.

2) It suffices to replace $F(H)$ by C_1 in the above proof.

Remark 3.2 *Note that all the results in this note still hold if we consider instead of the elementary operators $\Delta_{A,B}$ the elementary operators $\delta_{A,B}(X) = \sum_{i=1}^{n} A_i X B_i$.*

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ON SPACES X DETERMINED BY THE RINGS $C_k(X)$ AND $C_\infty(X)$

S. K. ACHARYYA AND S. K. GHOSH

ABSTRACT : Let $C_\infty(X)$ be the ring of all real valued continuous functions on a locally compact Hausdorff space X which vanish at infinity and $C_k(X)$ be its subring of all those functions which have compact support. We have shown that X is a P-space if and only if every prime ideal of $C_k(X)$ is maximal and in case $C_k(X) \neq C_\infty(X)$, there always exist non-maximal prime ideals in $C_\infty(X)$. Next we have shown that $C_\infty(X)$ is an ideal of $C(X)$ if and only if X is pseudo-compact. Finally we have established that X is extremally disconnected if and only if every non-empty subset of $C_k(X)$ bounded above in $C(X)$ has its supremum in $C(X)$.

2000 mathematics Subject Classification : 54C40, 46E25.

Key words and Phrases : Locally compact spaces, Rings of continuous functions with compact support, Rings of continuous functions which vanish at infinity, P-spaces, Pseudo-compact spaces, Extremally disconnected spaces.

1. INTRODUCTION

For any Tychonoff space X let $C(X)$ and $C^*(X)$ stand for the ring of all real valued continuous functions on X and that of all bounded real valued continuous functions on X respectively. One of the fascinating problems in the area of rings of continuous functions is to determine the class of spaces X, whose function rings $C(X)$ (respectively $C^*(X)$) satisfy some natural algebraic conditions. The well known and much talked of class of P-spaces stemmed out of such query; a space X is called P-space if and only if every prime ideal in the ring $C(X)$ is maximal equivalently if and only if every zero set in X is open. On the other end a number of important topological properties of X can be described by the ring and lattice structure of the rings $C(X)$ and $C^*(X)$. Mention may be made of extremal disconnectedness and basic disconnectedness of a space X in this context. Indeed a space X was proved by Hewitt long time back to be extremally disconnected if and only if the lattice ordered ring $C(X)$ is conditionally complete in the sense that every non-empty subset of $C(X)$ bounded above in $C(X)$ has its supremum in $C(X)$ and this is the case if and only if the lattice ordered ring $C^*(X)$ is conditionally complete. In this article we have addressed similer problems by investigating on some aspects of the interplay existing between the topological structure of a locally compact Hausdorff space X and the algebraic ring and order structure of $C_k(X)$ and $C_\infty(X)$. Here $C_\infty(X)$ is the ring of all those f in $C(X)$ which vanish at infinity in the sense that for each $n \in N$, $\{x \in X : |f(x)| \geq \frac{1}{n}\}$ is compact and $C_k(X)$ consists of those members f of $C_\infty(X)$ which have compact support in the sense that $cl_X\{x \in X : f(x) \neq 0\}$ is compact. Some aspects pertaining to internal structures of those two rings have been investigated recently by Acharyya, Chattopadhyaya and Ghosh [1] who have established that a locally compact Hausdorff space X is homeomorphic to the set of all maximal ideals of $C_k(X)$ (respectively $C_\infty(X)$) equipped with the so-called hull kernel topology. From this result they have deduced that two locally compact Hausdorff spaces X and Y are homeomorphic if and only if the rings $C_k(X)$ and $C_k(Y)$ (respectively the rings $C_\infty(X)$ and $C_\infty(Y)$) are isomorphic. Our principal motivation for writing the present article is derived from the last mentioned result, which tells in some way that one can pursue the problem of interaction, likely to be existing between the topological structure of a locally compact Hausdorff space X and the algebraic ring and order structure of $C_k(X)$ and $C_\infty(X)$.

Now, let us describe how we have organized our paper. An ideal 'I' unmodified in the ring $C_k(X)$ or $C_\infty(X)$ will always stand for a proper ideal i.e. I is not the whole of $C_k(X)$ (or $C_\infty(X)$). We have first established in section 2 that a locally compact Hausdorff space X is a P-space if and only if the ring $C_k(X)$ is regular in the sense of Von Neumann. In fact we have proved the following result in which a number of conditions, some topological and most of them algebraic are seen to be equivalent.

Theorem A. For a locally compact Hausdorff space X, the following statements are equivalent:

(1) Every prime ideal of $C_k(X)$ is maximal.

(2) For each p in X, $O_p \cap C_k(X) = M_p \cap C_k(X)$, where $M_p = \{f \in C(X) : f(p) = 0\}$ and $O_p = \{f \in M_p : f$ *vanishes on a neighbourhood of* $p\}$.

(3) For each f in $C_k(X)$, the zero set $Z(f) = \{x \in X : f(x) = 0\}$ is open in X.

(4) Every Ideal of $C_k(X)$ is a Z-ideal.

(5) Every ideal of $C_k(X)$ is an intersection of prime ideals of $C_k(X)$.

(6) For all f, g in $C(X)$, the principal ideal in $C_k(X)$ generated by $f^2 + g^2$ is identical to the ideal (f, g) generated by $\{f, g\}$ in the same $C_k(X)$.

(7) $C_k(X)$ is a regular ring (in the sense of Von Neumann) i.e. given $f \in C_k(X)$, there is a $g \in C_k(X)$ such that $f = f^2 g$.

(8) Given $f \in C_k(X)$, each real valued continuous function on $X - Z(f)$ has an extension to a function in $C_k(X)$.

(9) Every ideal of $C_k(X)$ is an intersection of maximal ideals of $C_k(X)$.

(10) Every principal ideal of $C_k(X)$ is generated by an idempotent.

(11) X is a P-space. [Compare with 4J, [2]].

It is clear that if $C(X) \neq C^*(X)$, then $C^*(X)$ always contains non maximal prime ideals because $C^*(X)$ is isomorphic to the ring $C(\beta X)$, where βX is the Stone-Cech compactification of X and a compact P-space is finite. We have established an analogous fact here, as is manifested by the following proposition:

Theorem B. Let X be a locally compact Hausdorff space with $C_k(X) \neq C_\infty(X)$. Then there exist non maximal prime ideals in the ring $C_\infty(X)$.

$C_\infty(X)$ is known to be an ideal of $C^*(X)$ but not necessarily of $C(X)$ but if X is pseudo-compact then clearly $C_\infty(X)$ is an ideal of $C(X)$. We do not know whether the condition pseudo-compactness of X is necessary for this result in general. We have however established in the following proposition the necessity of pseudo-compactness on X, if X is assumed to be locally compact.

Theorem C. A locally compact non compact space X is pseudo-compact if and only if $C_\infty(X)$ is an ideal of $C(X)$.

It is easy to see that for any Tychonoff space X, the two rings $C_k(X)$ and $C_\infty(X)$ are sublattices of the lattice ordered ring $C^*(X)$. In section 3, we have established first the following theorems in which the property extremal disconnectedness of a locally compact Hausdorff space X has been shown to be equivalent to some lattice like conditions on $C_k(X)$ and $C_\infty(X)$.

Theorem D. The following three conditions are equivalent for a locally compact Hausdorff space X.

(1) X is extremally disconnected.

(2) Every subset of $C_k(X)$ bounded above in $C(X)$ has its supremum in $C(X)$.

(3) Every subset of $C_\infty(X)$ bounded above in $C(X)$ has its supremum in $C(X)$.

The theorems (A) – (D) stated above clearly reveal that in the study of some aspects of interaction existing between the topology of X and the algebraic structures of the rings of continuous functions over X and their subrings, the ring $C_k(X)$ plays the same role as that of $C(X)$ and the ring $C_\infty(X)$ that of $C^*(X)$ – a remark which we make for a locally compact Hausdorff space X. It would have been therefore nice and befitting to the above observations if the extremal disconnectedness of such spaces X were equivalent to the conditional completeness of the ring $C_k(X)$ or the conditional completeness of the ring $C_\infty(X)$. Never-theless we have observed the following facts in this connection.

Theorem E. Let X be a Tychonoff space (not necessarily locally compact).

(1) If X is extremally disconnected then the lattices $C_k(X)$ and $C_\infty(X)$ are conditionally complete.

(2) If X is non compact and at least one of the two lattices $C_k(X)$ and $C_\infty(X)$ is conditionally complete, then every compact open subset of X is extremally disconnected.

2. *P* TOPOLOGY VERSUS REGULARITY OF $C_K(X)$ AND $C_\infty(X)$.

We state the following results of [1], which we write down for our ready reference.

Theorem 2.1. A Tychonoff space X is locally compact if and only if $\{Z(f) : f \in C_k(X)\}$ is a base for the closed sets of X if and only if $\{Z(f) : f \in C_\infty(X)\}$ is a base for the closed sets of X.

Theorem 2.2. For a locally compact Hausdorff space X, every maximal ideal M of the ring $C_\infty(X)$ is fixed in the sense that $\bigcap_{f \in M} Z(f) \neq \phi$.

We write down also the following known result, given in exercise 7G of Gillman and Jerison's text [2].

Theorem 2.3. Let X^+ stand for the one-point compactification of a locally compact Hausdorff space X. Then the ring $C_\infty(X)$ is isomorphic to the ideal $M_\infty = \{g \in C(X^+) : g(\infty) = 0\}$ of $C(X^+)$ under the map : $M_\infty \to C_\infty(X)$, $g \to g|X$.

Proof of theorem A.

(1) ⇒ (2) : Suppose for some point $p \in X$, $O_p \cap C_k(X) \neq M_p \cap C_k(X)$. We choose a point $q \neq p$ in X. Then from theorem 2.1, there is an $f \in C_k(X)$ such that $f(q) \neq 0$ but $Z(f)$ is a neighborhood of p. This implies that $f \in O_p \cap C_k(X) - M_q \cap C_k(X)$. Therefore $M_p \cap C_k(X)$ is the only maximal ideal in $C_k(X)$ which contains $O_p \cap C_k(X)$. But $O_p \cap C_k(X)$ is a Z-ideal in $C(X)$ and is therefore equal to the interesection of all prime ideals of $C_k(X)$ which contain it [see [2], 0.18, page-7]. Hence there exists at least one non-maximal prime ideal of $C_k(X)$ containing $O_p \cap C_k(X)$ and the condition (1) becomes false then.

(2) ⇒ (3) : Suppose (2) holds. We choose $f \in C_k(X)$ and $p \in Z(f)$. Then $f \in M_p \cap C_k(X) = O_p \cap C_k(X)$, which means that $Z(f)$ is a neighborhood of p. Thus $Z(f)$ is open.

(3) ⇒ (4) : Let (3) hold and I be an ideal of $C_k(X)$. Suppose $f \in I$ and $g \in C_k(X)$ are such that $Z(f) = Z(g)$. Define $h : X \to R$ as follows :

$$h(x) = \frac{g(x)}{f(x)},\ x \in X - Z(f)$$

$= 0,\ x \in Z(f)$.

Then $h \in C_k(X)$. Thus $g = f.h$ and hence $g \in I$. Therefore I becomes a Z-ideal of $C_k(X)$.

(4) $\Rightarrow$ (5) : This is immediate in view of the fact that every Z0ideal of $C_k(X)$ is the intersection of all the prime ideals of $C_k(X)$ which contain it–a result which can be established by adapting the arguments in the proof of theorem 2.8 of Gillman-Jerison text [2] in an obvious manner.

(4) $\Rightarrow$ (6) is trivial and (6) $\Rightarrow$ (7) is also trivial.

(7) $\Rightarrow$ (1) is true because in any commutative regular ring (even without identity) every prime ideal turns out to be a maximal one.

(5) $\Rightarrow$ (7) is immediate since for any f in $C_k(X)$, f and f^2 belong to the same set of prime ideals.

(3) $\Rightarrow$ (8) : Let (3) hold. Let $f \in C_k(X)$ and $g \in C(X - Z(f))$. We define $h(x) = g(x)$, if $x \in X - Z(f)$

$= 0,\ x \in Z(f)$.

Then $h \in C_k(X)$ and is an extension of g.

(8) $\Rightarrow$ (7) : Let (8) hold. We choose $f \in C_k(X)$, then $\frac{1}{f} \in C(X - Z(f))$. Hence there exists $g \in C_k(X)$ such that $g|X - z(f) = \frac{1}{f}$. Clearly then $f = f^2 g$ and hence $C_k(X)$ becomes a regular ring.

The equivalence of the statements (1) and (9) follows from the equivalence of (1) and (5).

(7) $\Leftrightarrow$ (10) is a special case of a more general result which tells that a commutative ring is regular if and only if every principal ideal is generated by an idempotent.

(11) $\Rightarrow$ (1) : Let X be a P-space. Then for each $f \in C(X)$, $Z(f)$ is open; since the implication (3) $\Rightarrow$ (8) has alrealdy been realized above, the condition (1) follows immediately.

(1) $\Rightarrow$ (11) : Suppose X is a P-space. Then there exists an $f \in C(X)$ such that $Z(f)$ is not open and therefore there is a point p in $Z(f)$ such that $p \in cl_X(X - Z(f))$. Now from theorem 2.1, there is a $g \in C_k(X)$ such that $p \in X - Z(g)$. Since $C_k(X)$ is an ideal of $C(X)$ it follows that $h = f.g \in C_k(X)$. One can check without difficulty that $p \in Z(h) \cap cl_X(X - Z(h))$. This

indicates that $Z(h)$ is not open in X and therefore the condition (3) does not hold, equivalently the condition (1) does not hold.

Proof of the theorem B.

Since $C_k(X)$ is a free ideal of $C(X)$ [see 4D 3[2]], it is also a free ideal of the ring $C_\infty(X)$. On the other hand $C_k(X)$ being a Z-ideal of $C(X)$ is the intersection of all the prime ideals of $C(X)$ which contain it [vide theorem 2.8 [2]] and therefore $C_k(X)$ is the intersection of all the prime ideals of $C_\infty(X)$ which contain it and each of these prime ideals is necessarily free. A straight way application of theorem 2.2 yields that none of these prime ideals in $C_\infty(X)$ is maximal.

Proof of the theorem *C*.

Let the locally compact space X be not pseudo-compact i.e. $C^*(X)$ is a proper subset of $C(X)$. Then there exists a copy $Y = \{x_n : n \in w_0\}$ of N which is C-embedded in X in the sense that every function in $C(Y)$ has an extension to a function in $C(X)$ [see 1.21, page-20[2]]. It is plain that Y is an F_σ subset of X^+ so that $X^+ - Y$ becomes a G_δ subset of X^+ with $\infty \in X^+ - Y$. Since in a Tychonoff space if a G_δ set G contains a compact set K, then there is a zero set Z such that $K \subset Z \subset G$, it follows that there is a function F in $C(X^+)$ whose zero set $Z_{X^+}(F)$ satisfies the relation $\infty \in Z_{X^+}(F) \subset X^+ - Y$. Since $F \in M_\infty$, by theorem 2.3, the function $F_0 = F|X$ belongs to $C_\infty(X)$ and clearly it does not vanish anywhere on Y. Set $G_0(x) = \frac{1}{F_0(x)}$, $x \in Y$. Then $G_0 \in C(Y)$ and has therefore an extension to a G in $C(X)$ because Y is C-embedded in X. Since $G.F_0 = 1$ on Y it is immediate that $G.F_0 = 1$ on $cl_X Y$, therefore $\{x \in X : |(G.F_0)(x)| \geq 1\} \supseteq cl_X Y$. But since Y is a copy of N, which is further C-embedded in $cl_X Y$, it follows that $cl_X Y$ cannot be compact and accordingly $\{x \in X : |(G.F_0)(x)| \geq 1\}$ is also non-compact. Hence $G.F_0$ is a function in $C(X)$ which does not belong to $C_\infty(X)$ and as $G \in C(X)$ and $F_0 \in C_\infty(X)$ we conclude that $C_\infty(X)$ is not an ideal of $C(X)$.

3. EXTREMAL DISCONNECTEDNESS OF X VERSUS THE LATTICE STRUCTURES OF $C_K(X)$ AND $C_\infty(X)$

We recall that a space X is extremally disconnected if and only if the closure of every open set in X is open.

Proof of the theorem *D*.

(1) ⇒ (2) follows from the well known result that for an extremally disconnected space X, the lattice $C(X)$ is conditionally complete [see 3N6, page-52[2]].

(2) ⇒ (1) : Suppose (2) holds. Let $V \neq X$ be any non-void open subset of X. Set $B = \{f \in C_k(X) : f \leq 1, f(X - V) = \{0\}\}$. The result of theorem 2.1 ensures that $B \neq \phi$ and

by assumed condition (2), $g = \sup B$ exists in $C(X)$ and we note that $g \leq 1$. The same theorem 2.1 further tells us that given $x \in V$, there is an $f \in C_k(X)$ such that $0 \leq f \leq 1$, $f(x) = 1$ and $f(X - V) = \{0\}$. It is clear that $f \in B$ and so $f \leq g$, consequently $g(x) = 1$. Thus $g(V) = \{1\}$ and it can be verified easily that $g(X - cl_X V) = \{0\}$. Hence $X - cl_X V = Z_X(g)$, the zero set of g in X and consequently $cl_X V$ becomes open in X.

(3) $\Rightarrow$ (2) is trivial.

Proof of theorem *E*.

(1) Let X be extremally disconnected. Then the lattice $C(X)$ becomes conditionally complete. Choose any subfamily $\{f_\alpha : \alpha \in \Lambda\}$ of $C_k(X)$ for which there is an $f \in C_k(X)$ with $f_\alpha \leq f$ for each $\alpha \in \Lambda$. Now there exists g in $C(X)$ such that $g = \sup\limits_{\alpha\in\Lambda} f_\alpha$ and for each $\alpha \in \Lambda$, $f_\alpha \leq g \leq f$. Since every Z-ideal in $C(X)$ is absolutely convex, in particular convex, it follows that $C_k(X)$ is a convex ideal of $C(X)$. Hence from the last inequality it follows that $g \in C_k(X)$. This proves that $C_k(X)$ is conditionally complete.

Now the extremally disconnected space X is an F-space so that every ideal in the ring $C(X)$ becomes convex [see 14N4, page 215[2] and theorem 14.25 [2]]; furthermore the fact that X is an F-space implies that βX is an F-space [theorem 14.25[2]] so that every ideal in $C(\beta X)$ is convex. But there exists a lattice isomorphism between the rings $C^*(X)$ and $C(\beta X)$ from which we can therefore say that every ideal in the ring $C^*(X)$ is also convex in particular the ideal $C_\infty(X)$ is convex. Hence arguing analogously as above, we can prove that $C_\infty(X)$ is a conditionally complete lattice.

(2) Assume that for the non-compact space X, the lattice $C_k(X)$ is conditionally complete. Let Y be a compact open subset of X. To show that Y is extremally disconnected it is equivalent to showing that the lattice $C(Y)$ is conditionally complete. Let B be a non-void subset of $C(Y)$ bounded above in this ring. We will produce a supremum of B in $C(Y)$ and that will do. With each g in B define $g^* : X \rightarrow R$ as follows.

$$g^*(x) = 0,\ x \in X - Y$$
$$= g(x),\ x \in Y.$$

Then $g^* \in C^*(X)$ and is an extension of g. Let $B^* = \{g^* : g \in B\}$. Since Y is compact, each element of B has compact support in Y and consequently then every element of B^* has compact support in X, this means that $B^* \subset C_k(X)$. Since $C_k(X)$ is assumed to be conditionally complete and B^* is bounded above in $C_k(X)$, as follows from the above assumption that B is bounded above in $C(Y)$, there exists a supremum h of the family B^* in $C_k(X)$. Let $g = h/_Y$. Then it is quite clear that $g = \sup B$ in $C(Y)$.

If we assume that the lattice $C_\infty(X)$ is conditionally complete, then in view of the convexity of the ideal $C_k(X)$ in the ring $C_\infty(X)$ it turns out that $C_k(X)$ also becomes conditionally complete. Hence the theorem is completely proved.

We complete this article with the following two seemingly open problems.

Problem 1.

Whether for a locally compact Hausdorff space X the conditional completeness of the lattice $C_k(X)$ is equivalent to the conditional completeness of $C_\infty(X)$?

Problem 2.

Does there exist a locally compact Hausdorff space X, which is not extremally disconnected but for which the lattice $C_k(X)$ is conditionally complete?

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SUB PSEUDO ALGEBRAIC SPACES

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ABSTRACT : If T is a p-topology on X and Y is a non-empty subset of X, then $T' = \{A \cap Y : A \in T\}$ is defined to be a p-topology on Y. Finally sub p-topological space and sub p-a space are established and some of its properties are discussed here.

Keywords : Pseudo topological space, sub pseudo topological space and sub pseudo algebraic space are denoted by p-topological space, sub p-topological space and sub p-a space.

1. INTRODUCTION

The aim of this paper is to introduce the notion of pseudo algebraic space. A pseudo algebraic space is defined to be a non-empty set having two types of structures - a pseudo topological structure and a pseudo algebraic structure. We then introduce the notion of sub p-a spaces, keeping it in mind that notions of topological subspaces and subgroups go together.

Our special interest would be on Sub p-topological space. We have introduced a p-a structure on sub p-topological space. A sub p-topological space equipped with a p-a structure is called a sub p-a space.

2. PRELIMINARIES

Definition 2.1. Let X be a non-empty set and T a class of subsets of X such that

(i) $X \in T$

(ii) there exists an $A_0 \in T$ such that $A_0 \subseteq A$ for every $A \in T$.

(iii) any finite intersection of members of T is a member of T.

The class T is called a pseudo topology (p-topology) and the pair (X, T) is called a p-topological space. Where there is no scope for confusion, X may be simply called a p-topological space. The members of T are called pseudo open sets (p-open sets) of X. A set A_0 with the property (ii) is called a minimal p-open set. In a p-topological space, there is one and only one minimal p-open set. Therefore, a minimal p-open set is referred as the minimal p-open set.

Definition 2.2. A p-topological space (X, T) is said to have a pseudo algebraic structure (p-a structure) if there exists a pseudo algebraic function,

$\alpha : P^X \times P^X \to P^X$ (P^X is the power set of X)

satisfying the following conditions :

(i) $\alpha(\alpha(A, B), C) = \alpha(A, \alpha(B, C))$, A, B $C \in P^X$

(ii) $\alpha(A, B) \in T$ if $\alpha(A, B) = \alpha(B, A)$ for $A, B \in T$

(iii) if $A_1 \subseteq A$, $B_1 \subseteq B$ then $\alpha(A_1, B_1) \subseteq \alpha(A, B)$

(iv) $\alpha(A_0, A) = \alpha(A, A_0)$, $A \in P^X$, where A_0 is the minimal p-open set.

We say that the triple (X, T, α) is a p-topological space with a p-a structure α or simply pseudo algebraic space (p-a space).

Definition 2.3. A subset A in a p-a space (X, T, α) is called a normal set if

$\alpha(A, Y) = \alpha(Y, A) \ \forall \ Y \in P^X$.

Definition 2.4. The p-topology T of a p-a space (X, T, α) is said to be normal if every p-open set is normal and a p-a space is said to be normal if its p-topology is normal.

3. SUB PSEUDO TOPOLOGICAL SPACES

Now we establish the sub p-topological space.

Proposition 3.1. If T is a p-topology on X and Y is a non-empty subset of X, then

$T' = \{A \cap Y : A \in T\}$ is a p-topology on Y.

Proof. (i) Since $X \in T$ and $X \cap Y = Y$ therefore $Y \in T'$

(ii) Let A_0 be the minimal element of T, then

$B_0 = A_0 \cap Y$ is the minimal element of T'

(iii) If $B_1, B_2, \ldots\ldots, B_n$ are members of T',

then $B_i = A_i \cap Y$ for some $A_i \in T$.

Now $\bigcap_{i=1}^{n} B_i = \bigcap_{i=1}^{n} (A \cap Y) = A \cap Y$ where $\bigcap_{i=1}^{n} A_i = A$

$\therefore$ T' is a p-topology on Y.

Definition 3.1. The p-topology T' on Y defined above is called the relative p-topology on Y and the p-topological space (Y, T') is called a Sub p-topological space of (X, T).

Example 3.1. Let $T = \{\{1, 3\}, \{1, 3, 4\}, \{1, 3, 5\}, X\}$ be a p-topology on $X = \{1, 2, 3, 4, 5\}$ and let $Y = \{1, 3, 5\}$ be a subset of X. Then $T' = \{\{1, 3\}, Y\}$ is a sub p-topology on Y and the pair (Y, T') is sub p-topological space.

Proposition 3.2. Let (Y, T') be a Sub p-topological space of a p-topological space (X, T) where $T' = \{G \cap Y : G \in T\}$. If $A \subset Y$ is p-open in X, then A is also p-open in Y.

Proof. Since $A \subset Y$, we have $A = A \cap Y \in T'$.

and $A \in T$. Therefore A is p-open in X.

Also $A \in T'$ and hence A is also p-open in Y.

Example 3.2. Let $T = \{\{1\}, \{1, 2\}, X\}$ be a p-topology with $\{1\}$ as the minimal p-open set on $X = \{1, 2, 3\}$. Let $Y = \{1, 3\}$ be a subset of X. Then $T' = \{\{1\}, Y\}$ is a p-topology on Y. If $A = \{1\}$, $A \subset X$ then $\{1\} \subset Y$. Therefore, $\{1\}$ is p-open in X and $\{1\}$ is also p-open in Y.

Therefore, $\{1\}$ is p-open in X and $\{1\}$ is also p-open in Y.

Proposition 3.3. Let (Y, T') be a sub p-topological space of a p-topological space (X, T). Then Y is p-open in X iff every subset B of Y which is p-open in Y is p-open in X.

Proof. Let every subset B of Y which is p-open in Y be p-open in X. We show that Y is p-open in X. Since Y is p-open in Y, $Y \subseteq Y$ and $Y = Y \cap Y$, $Y \in T$. Therefore Y is p-open in X.

Conversely, let B be a subset of Y which is p-open in Y. We show that B is p-open in X. Since B is p-open in Y, there exists a subset G which is p-open in X such that $B = G \cap Y$, $G \in T$. Since G is p-open in Y and G is p-open in X, therefore, B is p-open in X.

4. SUB PSEUDO ALGEBRAIC SPACES

In this section we establish the sub p-a space.

Definition 4.1. A sub p-topological space (Y, T') is called a sub p-a space of a p-a space (X, T, α) with a p-a structure α' if α induces a p-a function α' on P^Y such that

(i) $\alpha'(A, B) = \alpha(A, B)$, $A, B \in P^Y$

(ii) $\alpha'(A, B) \in T'$ if $\alpha'(A, B) = \alpha(B, A)$, for $A, B \in T'$

and (iii) $\alpha'(A'_0, A) = \alpha(A, A'_0)$, $A \in P^Y$ where A'_0 is the minimal element of T'.

α' is said to be the natural algebraic structure on Y induced by α and T' is said to be the induced p-a topology on Y by T.

Example 4.1. Let (X, T, α) be a p-a space where

$\alpha : P^X \times P^X \rightarrow P^X$ such that

$\alpha(A, B) = A \cup B,\ A, B \in P^X$

Let (Y, T') be a sub p-topological space of (X, T) where

$T' = \{A \cap Y : A \in T\}$ and let

$\alpha' : P^Y \times P^Y \rightarrow P^Y$ such that

(i) $\alpha'(A', B') = \alpha(A', B') = A' \cup B'$ for $A', B' \in P^Y$

(ii)
$$\begin{aligned}\alpha(A', B') &= A' \cup B'\\ &= (A \cap Y) \cup (B \cap Y)\\ &= (A \cup B) \cap Y\\ &= (B \cup A) \cap Y\\ &= (B \cap Y) \cup (A \cap Y)\\ &= B' \cup A'\\ &= \alpha(B', A')\end{aligned}$$

$\therefore\ \alpha'(A', B') \in \mathrm{T}'$

(iii) $\alpha(A'_0, A') = A'_0 \cup A',\ A' \in P^Y$ and A'_0 is the minimal element of T'.
$$\begin{aligned}&= (A_0 \cap Y) \cup (A \cap Y)\\ &= (A_0 \cap A) \cup Y\\ &= (A \cap A_0) \cup Y\\ &= (A \cap Y) \cup (A_0 \cap Y)\\ &= A' \cup A'_0\\ &= \alpha(A', A'_0)\end{aligned}$$

$\therefore\ (Y, T', \alpha')$ is a sub p-a space.

Example 4.2. Let (G, T, α) be a p-a space where G is a group and T is a p-topology called usual p-topology consisting of all subgroups of G with identity subgroup of G as the minimal p-open set and p-a structure on G is as follows :

$\alpha : P^G \times P^G \to P^G$ such that

$\alpha(A, B) = AB$, $A, B \in P^G$ and AB means

the ordinary product of the complexes A, B.

Let (K, T') be a sub p-topological space of (G, T) where K is a subgroup of G and $T' = \{K \cap G_\lambda : G_\lambda \in T, \lambda \in \Lambda\}$.

Let $\alpha' : P^K \times P^K \to P^K$ such that

(i) $\alpha'(A', B') = \alpha(A', B') = A'B'$ for $A', B' \subset K$

(ii)
$$\begin{aligned}\alpha'(A', B') &= A'B' \\ &= (A \cap K)(B \cap K) \\ &= (AB) \cap K \in T'\end{aligned}$$

Since A is a subgroup of $G \Rightarrow A \in T$

and B is a subgroup of $G \Rightarrow B \in T$.

$AB \in T$ iff $AB = BA$

$\alpha(A, B) \in T$ iff $\alpha(A, B) = \alpha(B, A)$

$\alpha'(A', B') \in T$ iff $\alpha'(A', B') = \alpha'(B', A')$, $A', B' \in T'$

(iii)
$$\begin{aligned}\alpha'(A'_0, A') &= A'_0A', \; A' \in P^K \text{ and } A'_0 \text{ is the minimal element of } T' \\ &= (A_0 \cap K)(A \cap K) \quad A, A_0 \in T \\ &= (A_0A) \cap K \\ &= (AA_0) \cap K, \; AA_0 \in T \\ &= (A \cap K)(A_0 \cap K) \\ &= A'A'_0\end{aligned}$$

$\therefore$ (K, T', α') is a sub p-a space.

Definition 4.2. **(a)** A sub p-a space (Y, T', α') of a p-a space (X, T, α) is called a normal sub p-a space if Y is a normal set in (X, T, α).

i.e. $\alpha(A, A_1) = \alpha(A_1, A)$, $A \in T'$, $A_1 \in P^X$.

(b) A sub p-a space (Y, T', α') is said to be normal in itself if every member of T' is normal in (Y, T', α') i.e. if.

$\alpha'(A, B) = \alpha'(B, A)$, $A \in T'$, $B \in P^Y$.

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Example 4.3. Let (X, T) be a p-topological space and let (X, T, α) be a p-a space where

$\alpha : P^X \times P^X \rightarrow P^X$ such that

$\alpha(A, B) = A \cup B$, $A, B \in P^X$.

Let (Y, T', α') be a sub p-a space of (X, T, α) where

$T' = \{A \cap Y : A \in T\}$

and $\alpha' : P^Y \times P^Y \rightarrow P^Y$ such that

$\alpha'(A', B') = \alpha(A', B') = A' \cup B'$ for $A', B' \in P^Y$.

Then (Y, T', α') is a sub p-a space [Example 4.1]

Again (Y, T', α') is a normal sub p-a space as well as normal in itself.

Let Y be a subset of X.

$$\begin{aligned}\text{Then } \alpha'(A', Y) &= A' \cup Y \quad \forall A' \in P^X \\ &= Y \cup A' \\ &= \alpha(Y, A')\end{aligned}$$

$\therefore$ Y is a normal set in (X, T, α)

$\therefore$ (Y, T', α') is a normal sub p-a space.

Also let A' be any member of T'

$$\begin{aligned}\alpha'(A', B') &= A' \cup B', \ B' \in P^Y \\ &= B' \cup A' \\ &= \alpha'(B', A')\end{aligned}$$

$\therefore$ $\alpha'(A', B') = \alpha'(B', A')$ $\forall A' \in T'$, $B' \in P^Y$

$\therefore$ Every member of T' is normal in (Y, T', α')

$\therefore$ (Y, T', α') is a normal sub p-a space in itself.

Proposition 4.1. Let H be a subgroup of group G and let T be a p-toplogy called usual p-topology on G. Let α be a p-a structure on G such that $\alpha(A, B) = AB$, $A, B \in P^G$. Then (G, T, α) is a p-a space. Let α' be a p-a structure on sub p-topological space (H, T') of p-topological space (G, T) such that

$\alpha'(A', B') = \alpha(A', B') = A'B'$ for $A', B' \subset K$. Then (H, T', α') is a normal sub p-a space of (G, T, α) if and only if H is a normal subgroup of G.

Proof : [From example 4.3]

(H, T', α') is a normal sub p-a space of $(G, T, \alpha) \Leftrightarrow H$ is normal in (G, T, α)

$\Leftrightarrow \alpha(H, A) = \alpha(A, H), A \in P^G$

$\Leftrightarrow HA = AH,$

$\Leftrightarrow H$ is a normal subgroup of G.

Proposition 4.2. Let H be a subgroup of a group G and T be the usual p-topology on G. Let $\alpha(A, B) = AB$, $A, B \in P^G$.

Then (H, T, α) is a normal sub p-a space in itself if and only if every subgroup of H is a normal subgroup of H.

Proof. (H, T', α') is normal in itself

$\Leftrightarrow \alpha'(A, B) = \alpha'(B, A), A \in P^H, B \in T'$

$\Leftrightarrow AB = BA$

$\Leftrightarrow A(H \cap G_\lambda) = (H \cap G_\lambda)A$ where $B = H \cap G_\lambda$, $G_\lambda \in T$, $\lambda \in \Lambda$.

$\Leftrightarrow AH_\lambda = H_\lambda A$ where H_λ is any subgroup of H which is of the form $H \cap G_\lambda$.

$\Leftrightarrow H_\lambda$ is a normal subgroup of H.

Conclusion. The study of the sub p-a space is to be continued and many of its implications are to be exposed in future.

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ON FUZZY SUBSTRUCTURES OF NEAR-RINGS

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ABSTRACT : In this paper we investigate certain properties of fuzzy substructures of a near-ring N. Some interesting results are established using the concepts of fuzzy prime N-subgroup, fuzzy irreducible N-subgroup and fuzzy primary N-subgroup of N. A fuzzy subset is defined analogous to the radical of a fuzzy ideal of a ring and it is shown that the fuzzy subset is a fuzzy semiprime N-subgroup and some related properties are obtained.

Key words and phrases : Near-ring, near-ring group, fuzzy ideal, fuzzy primary N-subgroup, fuzzy prime N-subgroup.

INTRODUCTION

The concept of fuzzy sets was introduced by Zadeh [6]. Since then these ideas have been applied to other algebraic structures like groups, rings, modules, topologies and so on. The notions of fuzzy sub near-ring and ideals were introduced by *S.* Abou-Zaid in 1991[1]. In this paper, we investigate certain properties of fuzzy substructures of a near-ring N. Some interesting results are established using the concepts of fuzzy prime N-subgroup, fuzzy irreducible N-subgroup and fuzzy primary N-subgroup of N. It is proved that the intersection of all fuzzy prime N-subgroups of N containing a fuzzy N-subgroup is a fuzzy N-subgroup which is analogous to the radical of an ideal of a ring. The fuzzy irreducibility character of an N-subgroup leads to another fuzzy N-subgroup which is fuzzy prime. In some special cases we find that every fuzzy irreducible N-subgroup is fuzzy primary N-subgroup. Finally we show that a fuzzy N-subgroup can be expressed as the intersection of fuzzy irreducible N-subgroups and in certain cases the same can be expressed as the intersection of primary N-subgroups.

1. DEFINITIONS AND NOTATIONS

A right near-ring [5] N is a system with two binary operations "+" and "." such that (i) $(N,+)$ forms a group, not necessarily abelian, (ii) $(N,.)$ forms a semi group and (iii) $(x + y)z = xz + yz$, for all $x, y, z \in N$.

A near-ring N is said to be *zero symmetric* if $x.0 = 0$, for all $x \in N$.

Throughout our discussion near-ring N means a zero symmetric right near-ring with unity, $S(N)$ denotes the class of fuzzy subsets of N and $F(N)$ denotes the class of fuzzy left N-subgroups of N.

Let $\mu \in S(N)$. Then μ is called a *fuzzy left (right) N-subgroup* [1] of N if for $x, y \in N$, (i) $\mu(x-y) \geq \text{Min}\{\mu(x), \mu(y)\}$ and (ii) $\mu(xy) \geq \mu(y)$ $(\mu(xy) \geq \mu(x))$.

μ is called fuzzy *N-subgroup* if μ is both left and right fuzzy N-subgroup of N.

A fuzzy subset μ of N is called a *fuzzy left ideal* [1] of N if (i) μ is fuzzy subgroup of N, (ii) $\mu(x+y) = \mu(y+x)$, (iii) $\mu[a(b+x) - ab] \geq \mu(x)$, for a, b, x, y in N.

A fuzzy subset μ of N is called a *fuzzy right ideal* [1] of N if (i) μ is fuzzy subgroup of N, (ii) $\mu(x+y) = \mu(y+x)$, (iii) $\mu(xy) \geq \mu(x)$, for all x, y in N.

If μ is both fuzzy right as well as fuzzy left ideal of N then μ is called a *fuzzy ideal* of N.

Let μ be a fuzzy N-subgroup of N and $\mu_0 = \{x \in N | \mu(x) = \mu(0)\}$. If $\mu_0 \neq N$ then we say μ is a *proper fuzzy N-subgroup* of N.

Let $A \subseteq N$ and $t \in [0, 1]$. We define $t_A(x) = t$, if $x \in A$

$= 0$, if $x \notin A$.

Then t_A is a fuzzy subset of N. If $A = \{x\}$, then t_x is called a *fuzzy point* of N. Also 1_A is the characteristic function of A. Further $t_N(x) = t$, for all $x \in N$.

A left N-subgroup A of N is called *irreducible* [2] if $A = I \cap J$, I and J are left N-subgroups of N, then either $A = I$ or $A = J$.

Let $\mu \in F(N)$. Then μ is called a *fuzzy irreducible* N-subgroup if $\mu = \theta \cap \sigma$ implies $\mu = \theta$ or $\mu = \sigma$, where θ and σ are fuzzy N-subgroups of N.

A left N-subgroup A of N is said to be *primary N-subgroup* [2] if $ab \in A$ and $a \notin A$ then $b^n \in A$, for some $n \in Z_+$.

Let $\mu \in F(N)$. Then μ is called fuzzy *primary N-subgroup* of N if $\mu(ab) > \mu(a)$ implies $\mu(ab) \leq \mu(b^n)$, for all $a, b \in N$, for some $n \in Z_+$.

A fuzzy N-subgroup μ of a near-ring N is said to be *fuzzy prime N-subgroup* if μ is not a constant function and for any fuzzy N-subgroups σ, θ of N, $\sigma\theta \subseteq \mu$ implies $\sigma \subseteq \mu$ or $\theta \subseteq \mu$.

For any N-subgroup I of N we define

$\sqrt{I} = \{x \in N | x^k \in I$, for some positive integer $k\}$.

Let $\mu \in F(N)$. Then we define a fuzzy subset $\sqrt{\mu}$ of N such that

$\sqrt{\mu}(x) = \sup\ \{t \mid x \in \sqrt{\mu_t}\},\ t \in Im\mu$.

A fuzzy N-subgroup μ of N is called *fuzzy semiprime* if $\mu(n^k) = \mu(n)$, for all $n \in N$ and for all $k \in Z^+$.

Let A be an N-subgroup of N. An element $a \in N$ is said to be *prime* to A if $ab \in A$ implies $b \in A$, for every $b \in N$.

We denote $P_A = \{a \in N : \text{a is non prime to } A\}$

Let $\mu \in F(N)$. Then we define a fuzzy subset σ_μ of N such that

$\sigma_\mu(x) = \sup\{t \mid x \in P_{\mu_t}\},\ \text{t} \in \text{Im}\mu$

$= \mu(x)$, if $\text{x} \notin P_{\mu_t}$, for any $t \in Im\mu$.

2. PRELIMINARIES

Lemma 2.1. [1] *A* fuzzy subset μ of N is a fuzzy left N-subgroup if and only if the level subset μ_t, $t \in Im\mu$ is a left N-subgroup of N.

Lemma 2.2. If μ is a fuzzy irreducible N-subgroup of N then $|Im\mu| = 2$ and $1 \in Im\mu$.

Proof. If possible let $m, s, t \in Im\mu$ such that $m < s < t = \mu(0)$. We define fuzzy subsets σ and θ of N such that

$$\begin{aligned} \sigma(x) &= t, \text{ if } x \in \mu_t \text{ and} & \theta(x) &= t, \text{ if } x \in \mu_t \\ &= k, \text{ if } x \in \mu_s - \mu_t,\ s < k < t & &= s, \text{ if } x \in \mu_s - \mu_t \\ &= m, \text{ if } x \notin \mu_s & &= r, \text{ if } x \notin \mu_s,\ m < r < s \end{aligned}$$

It can be easily verified that σ and $\theta \in F(N)$. If $\mu(x) = t$ then $x \in \mu_t$ implies $\theta(x) = t$ and $\sigma(x) = t$. So, $\mu = \theta \cap \sigma$. In case $\mu(x) = s$ we have $x \in \mu_s$ and $x \notin \mu_t$. It follows $\theta(x) = s$ and $\sigma(x) = k$ which give $(\theta \cap \sigma) = s$ and thus $\mu = \theta \cap \sigma$. Finally if $\mu(x) = \text{m}$ then $x \notin \mu_s$ and it implies $\sigma(x) = m$, $\theta(x) = r$. Thus $\mu = \theta \cap \sigma$.

Hence in all the cases $\mu = \theta \cap \sigma$. But it is seen that $\mu \neq \theta$ and $\mu \neq \sigma$ which contradicts that μ is fuzzy irreducible. As μ is non constant we get the required result.

Next, let $\mu(0) = t < 1$. We define fuzzy subsets θ and σ of N such that $\sigma(x) = t^* > t$, if $x \in \mu_t$, $\sigma(x) = \mu(x)$ if $x \notin \mu_t$ and $\theta(x) = t$, for all $x \in N$. Then it is clear that $\theta, \sigma \in F(N)$. Also $\mu = \theta \cap \sigma$ while $\mu \neq \theta$, $\sigma \neq \mu$ contradicting irreducibility of μ. Thus $\mu(0) = 1$ and $1 \in Im\mu$.

Lemma 2.3. If μ is a non-constant fuzzy irreducible left N-subgroup of N then μ_0 is an irreducible left N-subgroup of N, where $\mu_0 = \{x \in N | \mu(x) = \mu(0)\}$.

Proof. Let $x, y \in \mu_0$. Then $\mu(x) = \mu(y) = \mu(0)$. Now $\mu(x - y) \geq \min\{\mu(x), \mu(y)\} = \mu(0)$. It gives $\mu(x - y) = \mu(0)$ and thus $x - y \in \mu_0$. Also let $x \in \mu_0$, $n \in N$. Then μ being a fuzzy N-subgroup of N we get $\mu(nx) \geq \mu(x) = \mu(0)$. It follows that $nx \in \mu_0$. Thus μ_0 is a left N-subgroup of N.

Next we prove the irreducibility of μ_0. If possible let A, B be two left N-subgroups of N such that $\mu_0 = A \cap B$ with $\mu_0 \neq A$, $\mu_0 \neq B$. It is obvious that $A \neq B$. We define two fuzzy subsets θ and σ of N such that

$\sigma(x) = \mu(0)$ if $x \in \mu_0$, $\sigma(x) = \frac{1}{2}(\alpha + \mu(0))$ if $x \in A - \mu_0$, $\sigma(x) = \alpha$, if $x \notin A$

$\theta(x) = \mu(0)$, if $x \in \mu_0$, $\theta(x) = \frac{1}{2}(\alpha + \mu(0))$, if $x \in B - \mu_0$, $\theta(x) = \alpha$, if $x \notin B$, where $Im\mu = \{1, \alpha\}$.

It is obvious that $\theta, \sigma \in F(N)$. If for $x \in N$, $\mu(x) = \mu(0)$ then $\sigma(x) = \mu(0)$ and $\theta(x) = \mu(0)$ which implies $\mu(x) = (\theta \cap \sigma)(x)$. Next if $\mu(x) = \alpha$, then $x \notin \mu_0$ and hence $x \notin A$ or $x \notin B$.

Thus either $\theta(x) = \alpha$ or $\sigma(x) = \alpha$ giving thereby $\mu = \theta \cap \sigma$ with $\mu \neq \theta$ and $\mu \neq \sigma$ which is not the case. Therefore μ_0 is irreducible left N-subgroup of N.

Lemma 2.4. If A is an irreducible N-subgroup of N then $\mu = \chi_A \cup \alpha_N$, $\alpha \in [0, 1]$ is a fuzzy irreducible N-subgroup of N.

Using lemma 2.2, lemma 2.3 and lemma 2.4 we get the following result.

Lemma 2.5. Let $\mu \in F(N)$. Then μ is fuzzy irreducible left N-subgroup of N if and only if $Im\mu = \{1, \alpha\}$, $\alpha \in [0, 1)$ and μ_t is a irreducible left normal N-subgroup of N, $t \in Im\mu$. The following can be easily obtained :

Lemma 2.6. Let A be a primary left N-subgroup of N. Then $\mu = \chi_A \cup \alpha_N$ is fuzzy primary N-subgroup of N.

As in [1], the following can be obtained.

Lemma 2.7. Let A $(\neq N)$ be a fuzzy N-subgroup of N. Then A is a prime N-subgroup of N if and only if χ_A is a fuzzy prime N-subgroup of N.

In a similar manner as in [3] we get the following.

Lemma 2.8. Let μ be a fuzzy prime N-subgroup of N. Then $|Im\mu| = 2$, $\mu(0) = 1$.

Lemma 2.9. [2] Let I be an N-subgroup of N and S be a multiplicative subset of N. If $I \cap S = \phi$, then there exists a prime N-subgroup P of N such that $I \subseteq P$ and $P \cap S = \phi$.

3. MAIN RESULTS

Now we prove our main results

Theorem 3.1. Let μ be a fuzzy N-subgroup of N. Then $\sqrt{\mu}$ is the intersection of all fuzzy prime N-subgroups of N containing μ and hence $\sqrt{\mu}$ is a fuzzy N-subgroup of N.

Proof. Let $X(\mu)$ be the set of all fuzzy prime N-subgroups of N containing μ.

Let $\eta = \bigcap_{\sigma \in X(\mu)} \sigma$. If $\sigma \in X(\mu)$ then $Im\sigma = \{1, \alpha\}$, $\alpha \in [0, 1)$. Let $x \in N$, $n \in Z^+$. If $\sigma(x^n) = 1$ then $x^n \in \sigma_0$ and σ_0 being prime we have $x \in \sigma_0$. Thus $\sigma(x) = 1$. And hence $\sigma(x^n) = \sigma(x)$. Also if $\sigma(x^n) = \alpha$ then $\alpha = \sigma(x^n) \geq \sigma(x)$ and so $\sigma(x) = \alpha$, as $|Im\sigma| = 2$. Thus $\sigma(x^n) = \sigma(x)$, for all $x \in N$ and for some positive integer n.

Hence $\sigma(x^n) = \sigma(x) \geq \mu(x^n)$, as $\mu \subseteq \sigma$, for every $\sigma \in X(\mu)$. This implies $(\cap\sigma)(x) \geq \sup \mu(x^n) = \sqrt{\mu}(x)$, for all $\sigma \in X(\mu)$. Thus $\eta(x) \geq \sqrt{\mu}(x)$ and hence $\sqrt{\mu} \subseteq \eta$.

Next if possible let $\sqrt{\mu} \subset \eta$. Then there exist $a \in N$ such that $\sqrt{\mu}(a) < \eta(a)$. Let $t \in [0, 1]$ such that $\sqrt{\mu}\,(a) < t < \eta(a)$. This gives $a^n \notin \mu_t$, for all positive integers n. Let $S = \{a, a^2, a^3, \ldots\}$. Then S is a multiplicative set such that $\mu_t \cap S = \phi$. By lemma 2.9, there exists a prime N-subgroup P of N such that $\mu_t \subseteq P$ and $P \cap S = \phi$. So $a \notin P$. We define a fuzzy subset ν of N such that $\nu = \chi_P \cup t_N$. Then ν is a fuzzy prime N-subgroup of N such that $\mu \subseteq \nu$. From above we get that $\nu(a) = t$, as $a \notin P$ and by assumption $\eta(a) > t$. But $\nu \in X(\mu)$ and $\nu(a) \geq \eta(a)$. This gives us a contradiction. Hence $\sqrt{\mu} = \eta$.

Theorem 3.2. Let μ be a fuzzy N-subgroup of N. Then $\sqrt{\mu}$ is a fuzzy semiprime N-subgroup of N.

Proof. Let $x \in N$, $n \in Z^+$. Then by definition $\sqrt{\mu}(x^k) = \sup \{t \mid x^k \in \sqrt{\mu_t}, t \in Im\mu\}$ and $\sqrt{\mu}(x) = \{t \mid x \in \sqrt{\mu_t}, t \in Im\mu\}$. Let $S_x = \{t \mid x \in \sqrt{\mu_t}, t \in Im\mu\}$ and $Sx^n = \{t \mid x^n \in \sqrt{\mu_t}, t \in Im\mu\}$. If $t \in Sx^n$ then $x^n \in \sqrt{\mu_t}$, $t \in Im\mu$ and so, there exists $m \in Z^+$ such that $x^{nm} \in \mu_t$. This gives that $x \in \sqrt{\mu_t}$. Thus $Sx^n \subseteq S_x$ and it implies $\sup Sx^n \leq \sup S_x = \sqrt{\mu}(x)$. Hence $\sqrt{\mu}(x^n) = \sqrt{\mu}(x)$.

Theorem 3.3. Let μ and θ be two fuzzy N-subgroups of N. Then

(i) $\sqrt{\mu}(0) = \mu(0)$, (ii) $\mu \subseteq \sqrt{\mu}$ (iii) $\sqrt{\mu}(x) = \sup \mu(x^n)$, for some positive integer n.

(iv) $\sqrt{\sqrt{\mu}} = \mu$, (v) $\sqrt{\mu_t} \subseteq (\sqrt{\mu})_t$ and $\sqrt{\mu_t} = (\sqrt{\mu})_t$, if $Im\mu$ is finite, (vi) $\mu \subseteq \theta$ implies $\sqrt{\mu} \subseteq \sqrt{\theta}$, (vii) $\mu + \theta \subseteq \sqrt{\mu} + \sqrt{\theta}$,

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(viii) $\sqrt{\mu}\cap\sqrt{\theta}=\sqrt{\mu\theta}=\sqrt{\mu\cap\theta}$, (ix) $\sqrt{\mu}(x)$ = sup $\mu(x^n)$, for some positive integer n.

The Proof is straightforward and so we omit it.

Theorem 3.4. Let μ be a proper fuzzy N-subgroup of N. Then

(i) $\sigma_\mu(0)=\mu(0)$, (ii) $(\sigma_\mu)_0=P_{\mu_0}$, (iii) $\mu\subseteq\sigma_\mu$, (iv) $(\sigma_\mu)_t\subseteq P_{\mu_t}$, $t\in Im\mu$.

Proof. Since $\mu_0\neq N$ and 0 is non prime to μ_0, by definition, (i) follows.

(ii) Let $x\in(\sigma_\mu)_0$. Then $\sigma_\mu(x)=\sigma_\mu(0)=\mu(0)$. This implies $x\in P_{\mu_0}$. Also, if $x\in P_{\mu_0}$ then $\sigma_\mu(x)=\mu(0)=\sigma_\mu(0)$. It gives $x\in(\sigma_\mu)_0$. Hence $P_{\mu_0}=(\sigma_\mu)_0$.

(iii) Let $\mu(x)=t$ such that $\mu_t\neq N$. Now $x\in\mu_t$ implies $x\in P\mu_t$, $\sigma_\mu(x)\geq t$.

Hence, $\sigma_\mu(x)\geq\mu(x)$.

Next let $\mu(x)=t$ such that $\mu_t=N$. If $x\notin P\mu_s$, for all $s\in Im\mu$, then $\sigma_\mu(x)=\mu(x)$. And if $x\in P\mu_s$, for some $s\in Im\mu$ then $s\geq t$. Thus for every $x\in N$, $\sigma_\mu(x)\geq\mu(x)$.

(iv) Let $x\in P\mu_t$. Then $\sigma_\mu(x)\geq t$. So, $x\in(\sigma_\mu)_t$. This proves the result.

Theorem 3.5. If μ is a non constant fuzzy irreducible N-subgroup of N then σ_μ is a fuzzy prime N-subgroup of N.

Proof. Clearly $\mu_0\neq N$, $|Im\mu|=2$ and by lemma 2.3, μ_0 is an irreducible N-subgroup of N. As a consequence of lemma 9[2], it follows that $P\mu_0$ is a prime N-subgroup of N. Hence $(\sigma_\mu)_0$ is a prime N-subgroup of N. Also $1\in Im\sigma_\mu$, for if $\sigma_\mu(0)=s<1$, then $\mu(0)=s<1$ which is not the case. By definition, $|Im\sigma_\mu|=2$. Thus σ_μ is a fuzzy prime N-subgroup of N.

Theorem 3.6. If every fuzzy prime N-subgroup is a minimal prime then every fuzzy irreducible N-subgroup is fuzzy primary.

Proof. Let μ be a fuzzy irreducible N-subgroup of N. Then σ_μ is a fuzzy prime N-subgroup of N. By theorem 3.1, $\sqrt{\mu}\subseteq\sigma_\mu$. Also we note that $|\mathrm{Im}\mu|=2$ and $|\mathrm{Im}\sigma_\mu|=2$. By lemma 10[2], $(\sigma_\mu)_0=P\mu_0=\sqrt{\mu_0}=(\sqrt{\mu})_0$. Hence $\sqrt{\mu}=\sigma_\mu$.

Next let a, $b\in N$ such that $\mu(ab)>\mu(a)$. Then $\mu(ab)=1$ gives $ab\in\mu_0$ and $a\notin\mu_0$. Thus $b\in P\mu_0$ and it implies that $\sigma_\mu(b)=\mu(0)$. It implies that $\sqrt{\mu}(b)=\mu(0)$ and so, $b\in\sqrt{\mu_0}$. Therefore we get $b^n\in\mu_0$, for some positive integer n. Thus $\mu(ab)=\mu(b^n)$ and it proves that μ is a fuzzy primary N-subgroup of N.

Theorem 3.7. Let μ be a fuzzy N-subgroup of N and $Im\mu=\{1,\alpha\}$, $\alpha\in[0,1)$. Then μ can be expressed as the intersection of all fuzzy irreducible N-subgroups of N containing μ.

Proof. If is obvious that $\mu_0 \neq N$. Then μ_0 is the intersection of all irreducible N-subgroups I_k, ($k = 1, 2, 3, ...$) of N such that $\mu_0 \subseteq I_k$, for all k. We define $\mu_k = \chi_{I_K} \cup \alpha_N$. Then μ_k is a fuzzy irreducible N-subgroup of N with $\mu \subseteq \mu_k$, for all k. Also $\mu = \cap\, \mu_k$.

Also let θ be any fuzzy irreducible N-subgroup of n such that $\mu \subseteq \theta$ and $Im\theta = \{1, \beta\}$. It is easy to get that $\alpha \leq \beta$ and $\theta_0 = I_k$, for some k. Thus we get $\mu = (\cap\mu_k)\cap\theta$, for all θ defined as above. This proves the result.

Theorem 3.8. If N satisfies ascending chain condition (acc) on its N-subgroups then every fuzzy N-subgroup μ with $Im\mu = \{1, \alpha\}$, $\alpha \in [0, 1)$ can be expressed as a finite intersection of primary fuzzy N-subgroups.

Proof. As $\mu_0 \neq N$, there exist primary N-subgroups $P_1, P_2, ..., P_m$ such that $\mu_0 = \bigcap_{k=1}^{m} P_k$.

Let $\mu_k = \chi_{P_K} \cup \alpha_N$ for $k = 1, 2, ..., m$. Then each μ_k is a fuzzy primary N-subgroup of N and $\mu = \bigcap_{k=1}^{m} \mu_k$.

Theorem 3.9. If every fuzzy prime N-subgroup is minimal prime then every fuzzy N-subgroup is the intersection of all primary N-subgroups that contains it.

The proof follows from theorem 3.7 and 3.8.

Theorem 3.10. If N satisfies the acc on its N-subgroups then for any $\mu \in F(N)$, $(\sqrt{\mu})^n \subseteq \mu$ for some $n \in Z^+$, where $Im\mu = \{1, \alpha\}$, $\alpha \in [0, 1)$.

Proof. Let $x \in N$. If $x \in (\sqrt{\mu_0})^n$, then for every $x = \prod_{i=1}^{m} x_i$, we have $x_i \in \sqrt{\mu_0}$, for all $i = 1, 2, ..., m$. This gives, $\sqrt{\mu}(x_i) = 1$, for all $i = 1, 2, ..., m$.

Hence, $(\sqrt{\mu})^n(x) = \sup\{\min\{\sqrt{\mu}(x_1), \sqrt{\mu}(x_2), ... \sqrt{\mu}(x_k)\}\} = 1$.

As $\mu_0 \neq N$, $(\sqrt{\mu_0})^n \subseteq \mu_0$. This gives that $\mu(x) = 1$. And so, $(\sqrt{\mu})^n(x) = \mu(x)$ if $x \in (\sqrt{\mu_0})^n$.

Next if $x \notin \left(\sqrt{\mu_0}\right)^n$ then for every expression $x = \prod_{i=1}^{m} x_i$ there exists i such that $x_i \notin \sqrt{\mu_0}$. Therefore, $(\sqrt{\mu})^n(x) = \alpha$ while $\mu(x) = 1$ or α. In whatever case may be, $(\sqrt{\mu})^n(x) \leq \mu(x)$, for all $x \in N$. Thus the result follows.

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BOOLEAN ALGEBRAS AND A*-ALGEBRAS

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ABSTRACT : This paper studies the algebraic structure of A*-algebra which is generated by a Boolean algebra and a new set of minimum number of axioms for Boolean algebra.

Key words : A*-algebra; Stone type representation; Boolean Algebra.

0. INTRODUCTION

It is well known that for any set X, 2^X, as a set $2 = \{0, 1\}$, is a Boolean algebra and that any Boolean algebra can be imbedded in 2^X for some set X. We consider the algebraic structure of 3^X, where $3 = \{0, 1, 2\}$ for some set X.

In a drafted paper "The Equational theory of Disjoint Alternatives", around 1989, E. G. Manes introduced the concept of ada which however differs from the definition of the ada [6]. While the ada of the earlier draft seems to be based on Boolean algebras, the latter concept is based on C-algebras introduced by Fernando Guzman and Craig C. Squir [2].

In 1994, P. Koteswara Rao in his thesis [4] firstly introduced the concept of A*-algebras and studied their equivalence with Adas [5], C-algebras [2], and Adas [6] and their connections with 3-rings, Stone type representations and introduced the concept of A*-clones and the if-then-else structures over A*-algebras and ideals of A*-algebras.

In this paper, we concentrate on Boolean algebras, a new set of minimum number of axioms for Boolean algebras using Huntington's theorem, A*-algebras and the methods of generating A*-algebras from Boolean algebras.

First we start with the concept of Boolean algebra.

1. BOOLEAN ALGEBRA

Definition 1.1. A Boolean algebra is an algebra $(B, \wedge, \vee, (-)', 0, 1)$ with two binary operations, one unary operation (called complementation), and two nullary operations which satisfies :

(1) $(B, \wedge, \vee)$ is a distributive lattice; (2) $x \wedge 0 = 0$, $x \vee 1 = 1$ for all $x \in B$; (3) $x \wedge x' = 0$, $x \vee x' = 1$ for all $x \in B$.

We can easily prove that $x'' = x$, $(x \vee y)' = x' \wedge y'$, $(x \wedge y)' = x' \vee y'$ for all $x, y \in B$.

Definition 1.2. Let X be a set. The Boolean algebra of subsets of X, $P(X)$, has as its universe $P(x)$ and as operations $\cup$, $\cap$, $'$, φ, X. The Boolean algebra $2 = \{0, 1\}$ is given by $(2, \vee, \wedge, ', 0, 1)$, where $(2, \vee, \wedge)$ is a 2 element lattice with $0 < 1$ and where $0' = 1$, $1' = 0$.

Alternative systems of postulates for Boolean algebras were intensively studied during the decades 1900-1940. E. V. Huntington wrote an influential early paper [3] on this subject. No attempt will be made here to survey the extensive literature on such postulate systems. We present here Huntington's postulates in 1.3. We present a new set of postulates of our own for Boolean algebra in 1.4.

Huntington's Theorem 1.3. [1] Let B have one binary operation $\vee$ and one unary operation $(-)'$ and define (i) $a \wedge b = (a' \vee b')'$ for all $a, b \in B$. Suppose for all $a, b, c \in B$, (ii) $a \vee b = b \vee a$; (iii) $a \vee (b \vee c) = (a \vee b) \vee c$ and (iv) $(a \wedge b) \vee (a \wedge b') = a$. Then B is a Boolean algebra.

Theorem 1.4. [7] Let B have one binary operation $\wedge$ and one unary operation $(-)'$ and define (i) $a \vee b = (a' \wedge b')'$ for all $a, b \in B$. Suppose for all $a, b, c \in B$, (ii) $a \vee b = b \vee a$; (iii) $(a \vee b) \vee c = a \vee (b \vee c)$; (iv) $(a \wedge b) \vee (a \wedge b') = a$. Then B is a Boolean algebra. We now introduce the concept of A*-algebra.

2. A*-ALGEBRA

Definition 2.1. [4] An algebra $(A, \wedge, *, (-)^{\sim}, (-)_{\pi}, 1)$ is an A*-algebra if it satisfies : For $a, b, c \in A$, (i) $a_{\pi} \vee (a_{\pi})^{\sim} = 1$, $(a_{\pi})_{\pi} = a_{\pi}$ where $a \vee b = (a^{\sim} \wedge b^{\sim})^{\sim}$;

(ii) $a_{\pi} \vee b_{\pi} = b_{\pi} \vee a_{\pi}$; (iii) $(a_{\pi} \vee b_{\pi}) \vee c_{\pi} = a_{\pi} \vee (b_{\pi} \vee c_{\pi})$; (iv) $(a_{\pi} \wedge b_{\pi}) \vee (a_{\pi} \wedge (b_{\pi})^{\sim}) = a_{\pi}$; (v) $(a \wedge b)_{\pi} = a_{\pi} \wedge b_{\pi}$, $(a \wedge b)^{\#} = a^{\#} \vee b^{\#}$ where $a^{\#} = (a_{\pi} \vee a^{\sim}{}_{\pi})^{\sim}$; (vi) $a^{\sim}{}_{\pi} = (a_{\pi} \vee a^{\#})^{\sim}$, $a^{\sim\#} = a^{\#}$; (vii) $(a * b)_{\pi} = a_{\pi}$, $(a * b)^{\#} = (a_{\pi})^{\sim} \wedge (b^{\sim}{}_{\pi})^{\sim}$; (viii) $a = b$ if and only if $a_{\pi} = b_{\pi}$, $a^{\#} = b^{\#}$. We write 0 for $1^{\sim}$, 2 for $0 * 1$.

Example 2.2. With the operations defined below $(3, \wedge, *, (-)^{\sim}, (-)_{\pi}, 1)$ is an A*-algebra where $3 = \{0, 1, 2\}$.

$\wedge$	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

(a)

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

(b)

$*$	0	1	2
0	0	2	2
1	1	1	1
2	0	2	2

(c)

x	0	1	2
$x^{\sim}$	1	0	2
x_{π}	0	1	0
$x^{\#}$	0	0	1

(d)

Verification of example 2.2. Since; $0_\pi = 0$, $1_\pi = 1$, $2_\pi = 0$, $0^\sim = 1$, $1^\sim = 0$ and from tables (a) and (b), we have $a \vee b = (a^\sim \wedge b^\sim)^\sim$ and (i), (ii), (iii), (iv) and (v) $(a \wedge b)_\pi = a_\pi \wedge b_\pi$ for all $a, b, c \in 3$. Since, $0^\# = 0$, $1^\# = 0$, $2^\# = 1$ and from table (c),

we have $a^\# = (a_\pi \vee a^\sim_\pi)^\sim$ and (v) $(a \wedge b)^\# = a^\# \vee b^\#$, (vi) and (vii) for all $a, b \in 3$.

And since $a_\pi \neq 1_\pi$, $0^\# \neq 2^\#$ and $1_\pi \neq 2_\pi$ we have (viii) $a = b$ if and only if $a_\pi = b_\pi$, $a^\# = b^\#$ holds in $3 = \{0, 1, 2\}$.

Note 2.3. 1. From 2.1 (i) to 2.1 (iv) and by theorem 1.3 $\mathcal{B}(A) = \{a_\pi / a \in A\}$ is a Boolean algebra with $\wedge$, $\vee$, $(-)^\sim$, 0 and $a \in \mathcal{B}(A) \Rightarrow a_\pi = a$. Since 1, 0, $(a_\pi)^\sim \in \mathcal{B}(A)$, we have $1_\pi = 1$, $0_\pi = 0$, $(a_\pi)^\sim_\pi = (a_\pi)^\sim$ and $a_\pi \wedge a^\# = 0$, $a * 0 = a_\pi$.

2[3]. Let $(B, \wedge, (-)', 0)$ be a Boolean algebra. Then,

$A(B) = \{(a_1, a_2) \mid a_1, a_2 \in B \text{ and } a_1 \wedge a_2 = 0\}$ becomes an A*-algebra with $1 = (1, 0)$, $0 = (0, 1)$ and for all $a, b \in A(B)$ where, $b = (b_1, b_2)$, $b_1, b_2 \in B$, $b_1 \wedge b_2 = 0$.

(i) $a \wedge b = (a_1b_1, a_1b_2 + a_2b_1 + a_2b_2)$ (where juxtaposition, +, $(-)'$ respectively $\wedge$, $\vee$, $(-)'$ in Boolean algebra B).

(ii) $a \vee b = (a_1b_1, a_1b_2 + a_2b_1, a_2b_2)$.

(iii) $a^\sim = (a_2, a_1)$

(iv) $a_\pi = (a_1, a_1^\sim)$

(v) $a * b = (a_1, a_1^\sim b_2)$

3. Let $(A, \wedge, *, (-)^\sim, (-)_\pi, 1)$ be an A*-algebra. From 2.3 Theorem [3] we have $A \cong A(B(A))$.

Theorem 2.4. [7]. Suppose $(B, \wedge, \vee, (-)', 0, 1)$ is a Boolean algebra. Define, $\approx$ on $B \times B$ as $(a, b) \approx (c, d)$ if and only if $a = c$ and $a'b = c'd$. Then, (i) $\approx$ is an equivalence relation on $B \times B$; $< a, b > = \{(c, d) \in B \times B / (a, b) \approx (c, d)\}$, the equivalence class containing (a, b). Let $\mathcal{A}_B = B \times B / \approx = \{<a, b> / (a, b) \in B \times B\}$.

(ii) For every $<a, b> \in \mathcal{A}_B$, there exist $e, f \in B$ and $ef = 0$ such that $(e, f) \in <a, b>$ and (e, f) is unique. (iii) Define, $\wedge$, $\vee$, $*$, $(-)_\pi$, $(-)^\sim$, 1, 0, 2 on $\mathcal{A}_B$ as follows :

Assume that $\mathcal{A}_B = \{<a, b> / a, b \in B, ab = 0\}$.

(a) : $<a, b> \wedge <c, d> = <ac, ad + bc + bd>$ (where juxtaposition, +, respectively $\wedge$, $\vee$ in the Boolean algebra B);

(b) : $<a, b> \vee <c, d> = <ac + ad + bc, bd>$; (c) : $<a, b>^\sim = <b, a>$; (d) : $<a, b>_\pi = <a, a'>$; (e) : $<a, b> * <c, d> = <a, a'd>$; (f) : $1 = <1, 0>$, $0 = <0, 1>$, $2 = <0, 0>$.

Then $(\mathscr{A}_B, \wedge, \vee, *, (-)^{\sim}, (-)_{\pi}, 0, 1, 2)$ is an A*-algebra.

Proof. (i) Suppose $(a, b) \in B \times B$. Since $a = a$, $a'b = a'b$, implies, $(a, b) \approx (a, b)$.

Hence $\approx$ is reflexive. Suppose, $(a, b) \approx (c, d) \Rightarrow a = c$ and $a'b = c'd \Rightarrow c = a$, $c'd = a'b$.

Therefore, $(c, d) \approx (a, b) \Rightarrow \approx$ is symmetric.

Suppose $(a, b) \approx (c, d)$ and $(c, d) \approx (e, f) \Rightarrow a = c$, $a'b = c'd$ and $c = e$, $c'd = e'f \Rightarrow a = e$, $a'b = e'f \Rightarrow (a, b) \approx (e, f)$. Hence, $\approx$ is transitive.

Hence $\approx$ is an equivalence relation on $B \times B$.

Define, $<a, b> = \{(c, d) \in B \times B \,/\, (a, b) \approx (c, d)\}$, is the equivalence class containing (a, b). Also define, $\mathscr{A}_B = B \times B \,/\approx\, = \{<a, b> \,/\, (a, b) \in B \times B\}$.

(ii) Suppose $<a, b> \in \mathscr{A}_B$. Let $e = a$, $f = a'b$. Then $e, f \in B$, $ef = 0$ and $(e, f) \in\, <a, b>$.

Suppose $e_1, f_1 \in B$ & $e_1 f_1 = 0$ and $(e_1, f_1) \in\, <a, b>$.

Since $(e, f) \in\, <a, b>$ and $(e_1, f_1) \in\, <a, b> \Rightarrow (e, f) \approx (a, b)$ and $(e_1, f_1) \approx (a, b) \Rightarrow (e, f) \approx (e_1, f_1) \Rightarrow e = e_1$ and $e'f = e'_1 f_1 \Rightarrow e = e_1$ and $f = f_1$.

Therefore, (e, f) is unique. Clearly, $<a, b> \,=\, <e, f>$. Then $\mathscr{A}_B = \{<e, f>/e, f \in B, ef = 0\}$.

(iii) $1_{\pi} = \,<1, 0>_{\pi} \,=\, <1, 1'> \,=\, <1, 0> \,= 1$.

$<a, b>_{\pi} \,=\, <a, a'>$; $<a, b>_{\pi\pi} \,=\, <a, a'>_{\pi} \,=\, <a, a'> \,=\, <a, b>_{\pi}$.

Hence, $B(\mathscr{A}_B) = \{<a, b>_{\pi}/(a, b) \in B \times B\} = \{<a, a'> \,/\, a \in B\}$. Therefore, $B(\mathscr{A}_B) \cong B$.

Therefore, 2.1 (i) to (iv) hold in $\mathscr{A}_B$.

Consider, $(<a, b> \wedge <c, d>)_{\pi} \,=\, <ac, ad + bc + bd>_{\pi} \,=\, <ac, (ac)'> \,=\, <ac, a' + c'>$.

Therefore, $(<a, b> \wedge <c, d>)_{\pi} \,=\, <ac, a' + c'>$.

Consider, $<a, b>_{\pi} \wedge <c, d>_{\pi} \,=\, <a, a'> \wedge <c, c'> \,=\, <ac, ac' + a'c + a'c'> \,=\, <ac, ac' + a'>$

$=$ $(ac, c' + a')$. Therefore, $(<a, b> \wedge <c, d>)_{\pi} \,=\, <a, b>_{\pi} \wedge <c, d>_{\pi}$.

Consider, $<a, b>^{\#} = (<a, b>_{\pi} \vee <a, b>^{\sim}_{\pi})^{\sim} = (<a, a'> \vee <b, a>_{\pi})^{\sim} = (<a, a'> \vee <b, b'>)^{\sim}$

$= (\langle ab + ab' + a'b, a'b'\rangle)^{\sim} = (\langle a + a'b, a'b'\rangle)^{\sim} = (\langle a + b, a'b'\rangle)^{\sim} = (a'b', a + b)$.

Therefore, $\langle a, b\rangle^{\#} = \langle a'b', a + b\rangle$. Clearly, $\langle a, b\rangle^{\sim\#} = \langle a, b\rangle^{\#}$.

Now, $\langle a, b\rangle^{\sim} = \langle b, a\rangle$; $\langle a, b\rangle^{\sim}{}_{\pi} = \langle b, b'\rangle$.

Consider, $(\langle a, b\rangle_{\pi} \vee \langle a, b\rangle^{\#})^{\sim} = (\langle a, a'\rangle \vee \langle a'b', a + b\rangle)^{\sim}$

$= (\langle aa'b' + a(a + b) + a'a'b', a'(a + b)\rangle)^{\sim} = (\langle 0 + a + ab + a'b', a'b\rangle)^{\sim}$

$= (\langle a + a'b', a'b\rangle)^{\sim} = (\langle b', b\rangle)^{\sim}$ (since $ab = 0$) $= \langle b, b'\rangle$.

Hence, $\langle a, b\rangle^{\sim}{}_{\pi} = (\langle a, b\rangle_{\pi} * \langle a, b\rangle^{\#})^{\sim}$.

Since, $\langle a, b\rangle * \langle c, d\rangle = \langle a, a'd\rangle$; $(\langle a, b\rangle * \langle c, d\rangle)_{\pi} = \langle a, a'd\rangle_{\pi} = \langle a, a'\rangle$. Also, $\langle a, b\rangle_{\pi} = \langle a, a'\rangle$.

Therefore, $(\langle a, b\rangle * \langle c, d\rangle)_{\pi} = \langle a, b\rangle_{\pi}$.

Consider, $(\langle a, b\rangle * \langle c, d\rangle)^{\#} = \langle a, a'd\rangle^{\#} = \langle a'(a'd)', a + a'd\rangle = \langle a'(a + d'), a + a'd\rangle$

$= \langle a'd', a + d\rangle$. Therefore, $(\langle a, b\rangle * \langle c, d\rangle)^{\#} = \langle a'd', a + d\rangle$.

Consider, $\langle a, b\rangle^{\sim}{}_{\pi} \wedge \langle c, d\rangle^{\sim}{}_{\pi}{}^{\sim} = \langle a, a'\rangle^{\sim} \wedge \langle d, d'\rangle^{\sim} = \langle a', a\rangle \wedge \langle d', d\rangle$

$= \langle a'd', a'd + ad' + ad\rangle = \langle a'd', a'd + a\rangle = \langle a'd', d + a\rangle$.

Therefore, $\langle a, b\rangle^{\sim}{}_{\pi} \wedge \langle c, d\rangle^{\sim}{}_{\pi}{}^{\sim} = \langle a'd', a + d\rangle$.

Hence, $(\langle a, b\rangle * \langle c, d\rangle)^{\#} = \langle a, b\rangle^{\sim}{}_{\pi} \wedge \langle c, d\rangle^{\sim}{}_{\pi}{}^{\sim}$.

Suppose, $\langle a, b\rangle_{\pi} = \langle c, d\rangle_{\pi}$; $\langle a, b\rangle^{\#} = \langle c, d\rangle^{\#}$.

Then, $\langle a, b\rangle_{\pi} = \langle c, d\rangle_{\pi} \Rightarrow \langle a, a'\rangle = \langle c, c'\rangle \Rightarrow (a, a') \approx (c, c') \Rightarrow a = c, a'a' = c'c' \Rightarrow a = c$.

Also, $\langle a, b\rangle^{\#} = \langle c, d\rangle^{\#} \Rightarrow \langle a'b', a + b\rangle = \langle c'd', c + d\rangle \Rightarrow (a'b', a + b) \approx (c'd', c + d) \Rightarrow a'b' = c'd', (a'b')'(a + b) = (c'd')'(c + d) \Rightarrow a + b = c + d$.

Consider, $a + b = c + d \Rightarrow a'(a + b) = a'(c + d) = c'(c + d)$ (since $a = c$) $\Rightarrow a'b = c'd$ (since $a'b = b$, $c'd = d$, as, $ab = 0$, $cd = 0$).

Therefore, $a = c$, $a'b = c'd \Rightarrow (a, b) \approx (c, d) \Rightarrow \langle a, b\rangle = \langle c, d\rangle$.

Therefore, $\langle a, b\rangle_{\pi} = \langle c, d\rangle_{\pi}$, $\langle a, b\rangle^{\#} = \langle c, d\rangle^{\#} \Rightarrow \langle a, b\rangle = \langle c, d\rangle$.

Hence, $(\mathcal{A}_B, \wedge, \vee, *, (-)_{\pi}, (-)^{\sim}, 1)$ is an A*-algebra.

Therefore, every Boolean algebra B generates an A*-algebra $\mathcal{A}_B = B \times B / \approx$.

Corollary 2.5. Let $(A, \wedge, *, (-)^{\sim}, (-)_{\pi}, 1)$ be an A*-algebra. $B(A)$ is defined as in Note 2.3 1. Then, $A \cong A_{B(A)}$.

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CHARACTERIZATION OF GROUP STRUCTURES OF MODULES AND RINGS WITH A.C.C. ON ANNIHILATORS

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ABSTRACT : In this paper we deal with the group structure of a Goldie ring as well as that of what is termed as a so-called Goldie module. Here we find criteria for a group A such that there is a ring R in a given class of rings so that its additive group is isomorphic to A. In some special cases, the additive group structure of a Goldie ring gives rise to interesting results. In case of a Goldie module, if the ring attached with it satisfies some sorts of finiteness conditions on its annihilators, then the group takes some elegant form so far its decomposition is concerned. Some converses of these aspects in case of a Goldie ring have also been attempted.

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Key words : Goldie ring, Goldie module, Independent family of submodules, Annihilator ideal, Torsion subgroup, Torsion free subgroup, Divisible group, Quasi-cyclic group, Fully Goldie module.

1. INTRODUCTION

In 1988 [2] Chowdhury has introduced the notion of a new algebraic structure – the so called **Goldie module** and has obtained some interesting results analogous to those of a Noetherian module, viz., the decomposition of zero of such a module and the Artin-Rees Theorem for a Goldie module in some special cases. It is to be noted that the attached ring here need not be a Noetherian ring in general as assumed in some corresponding results in a Noetherian module.

On the other hand, the case, which studies the relation between the structure of a ring and that of its additive group, is another interesting aspect. This is motivated from two view-points.

First one is : for a given group A, to find all rings R whose additive group is isomorphic to A and the other is : for a given class of rings, to find the criteria for a group A such that there is a ring R in the given class whose additive group is isomorphic to A.

No satisfactory answer may be given as far as the first case is concerned. We shall deal with ring classes whose additive groups are rather restricted. A characterization of those groups is dealt with here in case of Goldie rings and Goldie modules.

Here our objective is to look at the group structure of a Goldie module M over R from this view-point. In certain aspect, as a special case, the additive group structure of a Goldie ring gives rise to some interesting results. When the attached ring R satisfies the ascending chain condition (a.c.c.) on annihilators, the group $(M, +)$ takes some elegant form so far its decomposition is concerned. Lastly some converses on Goldie rings have also been dealt with.

2. PRELIMINARIES

Throughout our discussion R will mean a ring not necessarily commutative and need not contain the unity 1, M is a right R-module and R^+ will mean the additive group structure of the ring R.

A family $\{M_\alpha \mid \alpha \in \Lambda\}$ of submodules of M is an **independent family** if for each $\alpha \in \Lambda$ we have $M_\alpha \cap (\sum_{\beta \neq \alpha} M_\beta) = 0$. The submodules M_α, $\alpha \in \Lambda$ **span** M if $M = \sum_{\alpha \in \Lambda} M$. The module M is the (internal) direct sum of its submodules M_α, $\alpha \in \Lambda$, written as $M = \bigoplus_{\alpha \in \Lambda} M_\alpha$, if $\{M_\alpha \mid \alpha \in \Lambda\}$ is an independent family of submodules spanning M. If p is a prime (number) then the set $\{m \in M \mid p^n m = 0, \text{ for some } n \in Z^+\}$ is called the **pth component** of $(M, +)$. Similarly pth component of a ring can be defined.

For some $S \subseteq R$, the **right (left) annihilator** is $r_R(S) = \{x \in R \mid Sx = 0\}$ $(1_R(S) = \{x \in R \mid xS = 0\})$. For a left ideal B of R, $T \subseteq M$, $Ann_M(B) = \{m \in M \mid mB = 0\}$ is a submodule of M and $Ann_R(T) = \{x \in R \mid Tx = 0\}$ is a right ideal of R. It is to be noted that if $M = R$, then $Ann_R(S) = 1_R(S)$ and $Ann_R(T) = r_R(T)$. Moreover if M is an (R, S) bi-module corresponding results in case of left R-module and right S-module may be obtained.

The following are some relevant easy results used in what follows.

If a is any element in a group $(G, +)$ and n $(\in Z) \neq 0$, we understand by what na means.

Lemma. 2.1. (i) Given any $n \in Z^+$ and any submodule N of M, the set $nN = \{nx \mid x \in N\}$ is a submodule of M.

(ii) Given any $n \in Z^+$, we have $nM \subseteq Ann_M(R[n])$ where $R[n] = \{x \in R \mid nx = 0\}$ is an ideal of R.

(iii) The set of torsion elements of M forms a submodule of M.

(iv) If p is a prime number, then the p^th^ component $T_p = \{x \in M \mid p^n x = 0$, for some $n \in Z^+\}$ is a submodule of M.

The following are the corresponding results in case of a ring (in particular).

Lemma. 2.2. (i) Given any $n \in Z^+$, the set $nR = \{nr \mid r \in R\}$ is an ideal of R.

(ii) Given any $n \in Z^+$, any left (right) ideal I of R, the set $nI = \{nr \mid r \in I\}$ is a left (right) ideal of R.

(iii) Given any $n \in Z^+$, the set $R[n] = \{x \in R \mid nx = 0\}$ is an ideal of R and $nR \subseteq 1_R(R[n])$.

In particular, for an ideal I of R, $nI \subseteq 1_R(I[n])$.

(iv) The set of torsion elements of R^+ forms an ideal of R.

(v) If p is a prime, then the p^{th} component $\{x \in R \mid p^n x = 0$, for some $n \in Z^+\}$ is an ideal of R.

If M has no infinite independent family of submodules, then M is said to be **finite dimensional.**

(vi) For $X \subseteq M$, $A \subseteq B \subseteq R$, we have $Ann_R(X) = Ann_R(Ann_M(Ann_R(X)))$ and $Ann_M(A) \supseteq Ann_M(B)$.

(vii) Let N be a subgroup of an abelian group G. Then G satisfies the d.c.c. (a.c.c.) on subgroups if and only if N and G/N satisfy the d.c.c. (a.c.c.) on subgroups.

Some important notions of group theory here are divisible groups, torsion subgroups, torsion free subgroups, bounded groups and periodic groups which may be found in any standard textbook. For the sake of completeness, we like to state the following theorem which is of considerable importance in what follows :

Baer's Theorem : If D is a divisible subgroup of an abelian group G and if K is a subgroup such that $K \cap D = 0$, then D has a direct complement M containing K.

Before presenting the main results of the paper, the definitions of some notions and relevant discussions are given for the sake of completeness.

A right module M over a ring R is a **Goldie module** over R or **Goldie (right) R-module** if

(i) M is finite dimensional

(ii) R satisfies the ascending chain condition (a.c.c.) on the annihilators of subsets of M in R (i.e., any collection of annihilators of subsets of M in R contains a maximal element). Similarly a **Goldie (left) R-module** can be defined.

Thus every right Goldie ring is a Goldie module over itself as a right R-module. Also every finite dimensional right module over a right Noetherian ring R is a Goldie module over R. Moreover it can be easily seen that if M is an Artinian right module over a right Artinian ring R, then M is a Goldie module over R. We also note that an (S, R) bi-module M, when M is right Noetherian and left Artinian, it is Goldie over R. Moreover, a submodule of Goldie module is clearly Goldie.

It is known that a homomorphic image of a right Goldie ring need not be a right Goldie ring (page 4, [5]). And it is easily seen that a quotient ring R/I is right Goldie if and only if the right R-module R/I is Goldie module. Thus homomorphic image of a Goldie module need not be Goldie. And this prompts us to make the following definition (as in case of a Goldie ring) : If every homomorphic image of a Goldie module M is Goldie, then M is a **fully Goldie module.**

For a subset S of R (left ideal S of R) $1_R(S)$ $(Ann_M(S))$ is a left ideal (submodule of M) of R. If it is minimal as a left annihilator ideal (annihilator submodule), then it may contain properly a subgroup of R^+ (of M). If it does not happen, then such a minimal left annihilator ideal (annihilator submodule) of R (of M) is also minimal or simple as subgroup of R^+ (of M). Then such a ring (module) will be called **a ring (a module) satisfying the annihilator minimum subgroup condition (amsc).** For example in $Z_{30}[x]$ the ideals are 0, $<2>[x]$, $<3>[x]$, $<5>[x]$, $<6>[x]$, $<10>[x]$, $<15>[x]$. Annihilators of these ideals are $Z_{30}[x]$, $<15>[x]$, $<10>[x]$, $<6>[x]$, $<5>[x]$, $<3>[x]$, $<2>[x]$ respectively. Thus $<15>[x]$ is minimal as annihilator ideal but the subgroup $\{0, 15\}$ is contained in $<15>[x]$.//

With group $(G, +)$ we consider $R = \text{End}\ (G, G)$ which can be given a ring structure. For a subgroup H of G we consider $A(H) = \{f \in R \mid f(H) = 0\}$. and it is a left ideal of R.

We confine our discussion to the case $[C]$: **Where the family of all subgroups of G is in one-one correspondence with the family $\mathscr{I}$ of the corresponding annihilators of subgroups of G.** We observe that in case of Z-module Z_6, Z_{12}, Z_{30} etc., the family of subgroups and the corresponding family of the annihilators of subgroups are in one-one correspondence. We discuss the group structure of those ring R in which the family of subgroups of R^+ is in one-one correspondence with the family of corresponding left annihilators of subgroups. So in such type of rings if R satisfies the a.c.c. on its left annihilators, then R^+ will satisfy the a.c.c. on its subgroups.

We know that for any $A \subseteq R$, $1_R(A) = 1_R(r_R(1_R(A)))$. As in this case when we consider a left R-module M if for $M' \subseteq M$ and $S \subseteq R$, we have $Ann_R(M') = 1_R(r_R(S))$, then we term **$Ann_R(M')$ as an R-alternating annihilator ideal of R.** In the case when we consider a right R-module M if for $M_1, M_2 \subseteq M$, we have $Ann_R(M_1) = r_R(Ann_R(M_2))$, then we term **$Ann_R(M_1)$ as an M-alternating annihilator ideal of R.**

We note that : if $A_1, A_2, \ldots \subseteq M$ (M as a left R-module) and $Ann_R(A_1) \supseteq Ann_R(A_2) \supseteq \ldots$ then we get $r_R(Ann_R(A_1)) \subseteq r_R(Ann_R(A_2)) \subseteq \ldots$ which gives $Ann_M(r_R(Ann_R(A_1))) \supseteq Ann_M(r_R(Ann_R(A_2))) \supseteq \ldots$ (M as a right R-module) and hence $Ann_R(Ann_M(r_R(Ann_R(A_1)))) \subseteq Ann_R(Ann_M(r_R(Ann_R(A_2)))) \subseteq \ldots$ (M as right R-module). If R satisfies the a.c.c. on annihilators of subsets of right R-module M in R, then there exists an $n \in Z^+$ such that $Ann_R(Ann_M(r_R(Ann_R(A_n)))) = Ann_R(Ann_M(r_R(Ann_R(A_{n+1})))) = \ldots$ (1). When every annihilator of subset of the right R-module M in R is M-alternating annihilator ideal of R, we get $Ann_R(M_n) = r_R(Ann_R(A_n))$, $\forall n \in Z^+$ and $M_n \subseteq M$. From (1), we get $Ann_R(Ann_M(Ann_R(M_n))) = Ann_R(Ann_M(Ann_R(M_{n+1}))) = \ldots$ which gives $Ann_R(M_n) = Ann_R(M_{n+1}) = \ldots$ or, $r_R(Ann_R(A_n)) = r_R(Ann_R(A_{n+1})) = \ldots$ —(2). Again when every annihilator of subset of the left R-module M in R is R-alternating annihilator ideal of R, we get $Ann_R(A_n) = 1_R(r_R(S_n))$, $\forall n \in Z^+$ and $S_n \subseteq R$. From (2), $r_R(1_R(r_R(S_n))) = r_R(1_R(r_R(S_{n+1}))) = \ldots$ which gives $r_R(S_n) = r_R(S_{n+1}) = \ldots$ So $1_R(r_R(S_n)) = 1_R(r_R(S_{n+1})) = \ldots$ Hence $Ann_R(A_n) = Ann_R(A_{n+1}) = \ldots$ Therefore R satisfies the d.c.c. on annihilators of subsets of the left R-module M in R.

Thus in an (R, R) bi-module M, if every annihilator of subset of the left R-module M is an R-alternating annihilator ideal of R also if every annihilator of subset of the right R-module M is an M-alternating annihilator ideal of R and R satisfies the a.c.c. on annihilators of subsets of the right R-module M in R, then R satisfies the d.c.c. on annihilators of subsets of the left R-module M in R. In this case we say that **M is a bi-module with R-M alternating annihilator condition.**

3. MAIN RESULTS

Now we give the main results of the paper in five sections. The first section consists of the results on **modules and rings with a.c.c. on annihilators.** The results on **additive group structure of a Goldie module and some corollaries** that follow from these in case of a Goldie module and a Goldie ring are in the second section. The third section contains some results on **Goldie modules without quasi-cyclic groups and some corollaries** that follow from these in case of Goldie rings. The fourth section has some independent results on **additive group structures of Goldie rings** and the last section contains **some converses.**

3.1. MODULES AND RINGS WITH A.C.C. ON ANNIHILATORS

Theorem 3.1.1. If R is a ring with the amsc satisfying the a.c.c. on right annihilators of subsets of R, then for any $m \in M$, there is an $\alpha \in Z^+$ such that $m\ \alpha\ I$ is a divisible subgroup for any right ideal I.

Proof. Since R satisfies the a.c.c. on right annihilators of subsets of R, it satisfies the d.c.c. on left annihilators. Hence the collection $\{1_R(I[n]) \,|\, n = 1, 2, \ldots\}$ has a minimal element say, $(1_R(I[\alpha])$ and also $1_R(I[\alpha]) = \alpha I$, for $\alpha I \subseteq 1_R(I[n])$ by amsc. It follows that for any $z \in Z^+$ and $B = \alpha I$, $zB \subseteq B$ which gives $zB = B$ as B is simple as group giving, thereby B is divisible. Now if $m \in M$, then $mI \subseteq M$ as mI is a submodule of M. Then $N = (mB) = m(zB) = z(mB) = zN$, for any $z \in Z^+$. Hence N $(= m\ \alpha\ I)$ is divisible subgroup of M

Theorem 3.1.2. If M is a torsion free with the amsc so that it satisfies the a.c.c. on its annihilators and I is a left ideal of R, then the annihilator of $I[s]$ is divisible (for $s \in Z^+$).

Proof. Since M satisfies the a.c.c. on its annihilators, R satisfies the d.c c. on annihilators of subsets of R in M. Hence the collection $\{Ann_M(I[n]) \,|\, n = 1, 2, \ldots\}$ has a minimal element (say) $Ann_M(I[s])$ and also $Ann_M(I[s])$ is a submodule of M. If $N = Ann_M(I[s])$, then for $s \in Z^+$ we have $sN \subseteq N = Ann_M(I[s])$. And by amsc, $sN = Ann_M(I[s])$. If $B = sN$, then for any $z \in Z^+$, we get $zB \subseteq B$ which gives $zB = B$. Therefore B is divisible. Now we claim that N is divisible. We have, for $n \in N$, $sn \in sN = B$. Since B is divisible, we get some $m \in Z^+$, $sx \in B$, $x \in N$ such that $sn = m(sx) \Rightarrow s(mx) = sn \Rightarrow s(mx-n) = 0 \Rightarrow mx = n$ (as M is torsion free). Thus N is divisible.//

Theorem 3.1.3. If R is torsion free with a.c.c. on its right annihilators and satisfies the amsc, then for any left ideal I of R, the group I^+ is divisible.

Proof. Since R satisfies the a.c.c. on its right annihilators, the collection $\{1_R(I[n]) \mid n = 1, 2, \ldots\}$ of left annihilators has a minimal element (say) $1_R(I[m])$ and also $1_R(I[m]) = mI$ by amsc. For any $z \in Z^+$, $B = mI$, we have $zB \subseteq B$ which gives $zB = B$. Therefore B^+ is divisible. Let $n \in Z^+$ and $i \in I$. Now we have $mi \in mI = B$. Since B^+ is divisible, we get some $mk \in B$, $k \in I$ such that $n(mk) = mi$ or, $m(nk\text{-}i) = 0$. The ring R being torsion free, it therefore follows that $nk = i$. Hence I^+ is divisible.//

Now we prove a sufficient condition for finite generation of R^+ in case of a nilpotent ring R with the conditions [c] and the amsc which establishes a relation with an (R, R) bi-module M with R-M alternating annihilator condition.

Theorem 3.1.4. If R satisfies the a.c.c. on annihilators of subsets of M as right R-module, then R^+ is finitely generated.

Proof. R being nilpotent, so $R^k = 0$, $R^{k-1} \neq 0$, for some $k \in Z^+$. Let C be subgroup of R^+ such that $R^{i+1} \subseteq C \subseteq R^i$ $(i = 1, 2, \ldots, k-1)$. Clearly C is an ideal of R. We claim that the maximum condition is satisfied by the subgroups of R^+ lying between R^i and R^{i+1} $(i = 1, 2, \ldots, k-1)$. For if not, then there exists a strictly ascending chain of subgroups of R (say) $I_1 \subseteq I_2 \subseteq \ldots$ (I) such that $R^{i+1} \subseteq I_j \subseteq R^i$ $(j = 1, 2, \ldots)$. Taking right annihilators of (I), we get $r_R(I_1) \supseteq r_R(I_2) \supseteq \ldots$ (II). Here $B_j = r_R(I_j)$ is an ideal and since $R^k = 0$ so $B_j^k = 0$. Now $B_j^k M = 0 \Rightarrow B_j B_j^{k-1} M = 0 \Rightarrow Bj \subseteq Ann_R(B_j^{k-1}M)$, $(\forall j)$. By hypothesis, $\{Ann_R(B_j^{k-1}M)\}$ has a minimal element (say) $Ann_R(B_n^{k-1}M)$. Since by amsc $Ann_R(B_n^{k-1}M)$ is simple as group, $B_n = Ann_R(B_n^{k-1}M)$ as $Bn \subseteq Ann_R(B_n^{k-1}M)$. Clearly, $B_n = B_{n+1}$ for $B_n = Ann_R(B_n^{k-1}M) \supseteq B_{n+1}$. Hence R satisfies the d.c.c. on its right annihilators and so by [c], R^+ satisfies the a.c.c. on its subgroups. Therefore for the chain (I), we have an $n \in Z^+$ such that $I_n = I_{n+1} = \ldots$ Thus the maximum condition is satisfied by the subgroups of R^+ lying between R^i and R^{i+1} $(i = 1, 2, \ldots, k-1)$. It follows that R^i/R^{i+1} satisfies the maximum condition on its subgroups for $i = 1, 2, \ldots, k-1$. Hence by Lemma 2.2 (vii), R^+ also satisfies the a.c.c. on subgroups and therefore R^+ is finitely generated.//

It is easy to see

Corollary 3.1.5. If R is nilpotent satisfying the a.c.c. on its right annihilators with the amsc together with the condition [c], then R^+ is finitely generated.//

Theorem 3.1.6. If R is nilpotent with the amsc satisfying the a.c.c. on its right annihilators, then R^+ satisfies the d.c.c. on subgroups.

Proof. R being nilpotent, $R^k = 0$, $R^{k-1} \neq 0$, for some $k \in Z^+$. If C is a subgroup of R^+ such that $R^{i+1} \subseteq C \subseteq R^i$ $(i = 1, 2, ..., k-1)$, then C is an ideal of R. We claim that the minimum condition is satisfied by the subgroups of R^+ lying between R^i and R^{i+1}. If not, then there exists a strictly descending chain of subgroups of R^+(say) $G_1 \supset G_2 \supset ...$ such that $R^{i+1} \subseteq Gj \subseteq R^i$ $(j = 1, 2, ...)$. But $R^k = 0$ gives $G_j^k = 0$ $(j = 1, 2, ...)$. Therefore $G_j \subseteq 1_R(G_j^{k-1})$ as $G_jG_j^{k-1} = G_j^k = 0$. As G_1, G_2, ... are ideals of R, so are also G_1^{k-1}, G_2^{k-1}, ... Therefore $\{1_R(G_j^{k-1})\}$ is a collection of left annihilator ideals of R, giving thereby a minimal element (say) $1_R(G_m^{k-1})$. So by amsc, $G_m = 1_R(G_m^{k-1})$. Therefore $G_m = G_{m+1} = ...$ etc. Hence the minimum condition is satisfied by the subgroups lying between R^i and R^{i+1} $(i = 1, 2, ..., k-1)$ and therefore the group R^i/R^{i+1} $(i = 1, 2, ..., k-1)$ satisfies the d.c.c. on its subgroups. Hence by Lemma 2.2 (vii), the group R^+ satisfies the d.c.c. on its subgroups.//

3.2. ADDITIVE GROUP STRUCTURES OF GOLDIE MODULES

Theorem 3.2.1. If R satisfies the a.c.c. on right annihilators of its subsets and M is a Goldie module such that M and R both satisfy the amsc, then the group M is a direct sum of a number of copies of the additive group of rational numbers, a number of copies of additive finite cyclic groups and a finite number of quasi-cyclic groups.

Proof. Since R satisfies the a.c.c. on its right annihilators so it satisfies the d.c.c. on its left annihilators. Hence the collection $\{1_R(R[p]) \mid p = 1, 2, ...\}$ has a minimal element, (say) $1_R(R[n])$. Again by Lemmas 2.2(i), 2.2(iii) and by amsc, we get $nR = 1_R(R[n])$ $(= B)$ (say). We have $zB \subseteq B$ (for $z \in Z^+$). Hence $zB = B$ and so the additive group B^+ of R is divisible. By Baer's Theorem, we have $R^+ = B^+ \oplus H$, for some subgroup H of R^+. We get $nH = 0$ (as $nR = B$). Again M being Goldie, R satisfies the d.c.c. on annihilators of subsets of R in M. Hence the collection $\{Ann_M(R[\alpha]) \mid \alpha = 1, 2, ...\}$ has a minimal element (say) $Ann_M(R[p])$. Now we have $pM = Ann_M(R[p])$ $(= D)$, by Lemmas 2.1(i), 2.1(ii) and by amsc. But $zD \subseteq D$, for any $z \in Z^+$. Hence $zD = D$, giving thereby D is divisible. We have by Baer's Theorem, $M = D \oplus I$, for some subgroup I of M. As $pM = D$, we get $pI = 0$ and so by Theorem II 1.d. [6], I is a direct sum of a number of copies of finite cyclic groups. We know by Theorem 23.1 [3] that a divisible group is a direct sum of quasi-cyclic group $C(p^\alpha)$ $(= \{r/p^n + Z \mid p$, a prime, $r, n \in Z\})$ and full rational groups. We claim that the number of quasi-cyclic subgroups $C(p^\alpha)$ which are direct summands of M is finite. Now if a $(\in D)$ is of order m (and so $ma = 0$), $r \in R$, we have $r = b + h$ where $b \in B^+$ and $h \in H$ as $R^+ = B^+ \oplus H$. Since B^+ is divisible, for $b \in B^+$ and for $m \in Z^+$ there exists $b' \in B^+$ such that $mb' = b$ ...(i). Again $a \in D$ and

D is divisible, so for $n \in Z^+$ there exists $d \in D$ such that $nd = a$ ...(ii). Now, $ar = a(b + h) = ab + ah = a(mb') + (nd)h$ [by(i) and (ii)] $= (ma)b' + d.(nh)=0.b'+d.0$ [$nH = 0$, so $nh = 0$, $\forall h \in H$] $= 0$. Thus the elements of finite orders in D are annihilated by the elements of R. Therefore the torsion part of D is contained in $Ann_M(R)$. And hence it is a submodule of M. Clearly $C(p^\alpha)$ is a subgroup and since its elements are annihilated by the elements of R, it follows that $C(p^\alpha)$ is a submodule of M. Also from above it follows that the sum $\oplus\ C(p^\alpha)$ is annihilated by R and we get thus $\oplus\ C(p^\alpha)$ is a finite sum, otherwise M will have an infinite direct sum of submodules, contradicting Goldie character of M.//

We easily see the following :

Corollary 3.2.2. In the Goldie module M where R satisfies the a.c.c. on right annihilators of subsets of R as in the above theorem 3.2.1, then the quasi-cyclic subgroups of M belong to the annihilator of R.

Corollary 3.2.3. If R be a right Goldie ring with the amsc, then R^+ is a direct sum of a number of copies of the additive group of rational numbers, a number of copies of additive finite cyclic groups and a finite number of quasi-cyclic groups.

Corollary 3.2.4. In the right Goldie ring R of Corollary 3.2.3, the quasi-cyclic subgroups of R belong to the annihilator of R.

3.3. GOLDIE MODULES WITHOUT QUASI-CYCLIC GROUPS

Theorem 3.3.1. If R satisfies the a.c.c. on its right annihilators, $1 \in R$, M is Goldie such that M and R both satisfy the amsc, then the group M is a direct sum of a number of copies of the additive group of rational numbers and a number of copies of additive finite cyclic groups.

Proof. As in Theorem 3.2.1, we get $M = D \oplus I$ where D is divisible and I is periodic subgroup. The group D, being divisible, is a direct sum of quasi-cyclic groups $\{r/p^n + Z \mid p \text{ is prime}, r, n \in Z\}$ and full rational groups. But quasi-cyclic subgroups of M belong to the annihilator of R, by Corollary 3.2.2. It follows that quasi-cyclic subgroups are trivial as $1 \in R$. Thus M is a direct sum of a number of copies of the additive group of rational numbers and a number of copies of additive finite cyclic groups.//

We easily see

Corollary 3.3.2. If R is a right Goldie ring, $1 \in R$ with the amsc, then R^+ is a direct sum of a number of copies of the additive group of rational numbers and a number of copies of additive finite cyclic groups.

Theorem 3.3.3. If R satisfies the a.c.c. on right annihilators of its subsets, $1 \in R$, M be fully Goldie module such that M and R both satisfy the amsc, then M is a direct sum of a torsion free Goldie module S and a finite number of Goldie modules T_{pi} belonging to different primes p_i (in other words, $M = S \oplus T_{p1} \ldots. \oplus T_{pn}$.)

Proof. As in Theorem 3.3.1, we have $M = S \oplus T$, where S is a number of copies of the additive groups of rational numbers and T is the torsion part of M. As this torsion part of M is a submodule of M and it is the direct sum of its p^{th} components T_{pi} which are submodules of M. The module M being Goldie, only a finite number of the p^{th} components are different from 0's, otherwise M will have an infinite direct sum of submodules, contradicting the Goldie character of M. Thus $M = S \oplus T_{p1} \oplus \ldots \oplus T_{pn}$ where $T = T_{p1} \oplus \ldots \oplus T_{pn}$. Then S is isomorphic to M/T, a homomorphic image of M and M being fully Goldie, it is again Goldie. Further it is torsion free. Since each of T_{pi} is a Goldie module so M is the module theoretic direct sum of torsion free Goldie modules and a finite number of Goldie modules T_{pi}.//

It is now easy to see

Corollary 3.3.4. If R is a fully right Goldie ring, $1 \in R$ and R satisfies the amsc, then R is a direct sum of a torsion free right goldie ring S and a finite number of right Goldie rings T_{pi} belonging to different primes p_i (in other words, $R = S \oplus T_{p1} \oplus \ldots \oplus T_{pn}$).

Theorem 3.3.5. If R satisfies the a.c.c. on its right annihilators and M is Goldie such that M and R both satisfy the amsc and annihilator of R is Noetherian as a right R-module, then M cannot have any quasi-cyclic subgroups.

Proof. As in Corollary 3.2.2, quasi-cyclic subgroups of M belong to the annihilator of R. If B is a subgroup of $Ann_M(R)$ and $b \in B$ ($\subseteq Ann_M(R)$), then $bR = 0 \subseteq B$ and so every subgroup of the annihilator $Ann_M(R)$ is a submodule of it. The annihilator $Ann_M(R)$ satisfies the a.c.c. on submodules, as $Ann_M(R)$ is Noetherian as a right R-Module. Now suppose M contains a quasi-cyclic subgroup. But we know that a quasi-cyclic subgroup does not satisfy the a.c.c. on its subgroups and every subgroup of $Ann_M(R)$ and hence a quasi-cyclic subgroup of M is a submodule of $Ann_M(R)$. Therefore $Ann_M(R)$ does not satisfy the a.c.c. on submodules which contradicts the Noetherian structure of $Ann_M(R)$.

Hence M cannot have any quasi-cyclic subgroups.//

3.4. ADDITIVE GROUP STRUCTURES OF GOLDIE RINGS

Theorem 3.4.1. If the left annihilator A of a right Goldie ring R be such that A satisfies the amsc and the a.c.c. on right annihilators of subsets of A and also the condition [c], then the group A^+ is finitely generated and A is a right Goldie ring.

Proof. We have $1_R(R) = A$. Now $A^2 = AA \subseteq AR = 0$. Thus A is nilpotent. If $S \subseteq A$, $AS(AA = A^2) = 0$ i.e., $A \subseteq 1_R(S)$. By hypothesis, $\{1_R(S)|S \subseteq A\}$ has a minimal element (say) $1_R(S)$. Since by amsc, $1_R(S)$ is simple as group, $A = 1_R(S)$ as $A \subseteq 1_R(S)$. Hence the minimum condition for left annihilator of subsets of A is trivially satisfied. If B is a right ideal of A, BR $(\subseteq AR = 0) \subseteq B$ and so B is a right ideal of R. Thus a right ideal of A is a right ideal of R also. And R being right Goldie, A cannot have an infinite direct sum of right ideals. Therefore A is a right Goldie ring and result follows from the Corollary 3.1.5.//

Theorem 3.4.2. If the left annihilator A of a right Goldie ring R be such that A satisfies the a.c.c. on right annihilators of subsets of A and also satisfies the amsc, then the group A^+ satisfies the d.c.c. on its subgroups and A is right Goldie ring.

Proof. As in Theorem 3.4.1, the left annihilator A of R is a right Goldie ring and is nilpotent. Therefore the result follows from Theorem 3.1.6.//

3.5. SOME CONVERSES

Theorem 3.5.1. If $(A,+)$ be an abelian group such that it is a direct sum $\oplus Q \oplus C(P_j^{Kj})$of a number of copies of the additive group of rational numbers and a number of copies of additive finite cyclic groups where $p_j^{kj}|m$, $(m \in Z)$, then there exists a Goldie ring R such that $R^+ = A$.

Proof. The first summand of A is a direct sum $\oplus Q$ of copies of the additive group Q of rationals and Q is a Goldie ring with 1. The second summand of A is a direct sum $\oplus C(P_j^{Kj})$, ($p_j^{kj}|m$, m is an integer) of number of copies of additive finite cyclic groups. Each $C(P_j^{kj})$ is a direct sum of finite cyclic groups of some prime power order p_j^{kj}. Now by Lemma 122.3 [4], there exists an associative and commutative ring with 1 on each $C(P_j^{Kj})$ with a finite numbers of ideals. Hence each of these rings is a Goldie ring with 1. Thus $\oplus C(P_j^{Kj})$ is a Goldie ring with 1. Therefore R is a Goldie ring with 1 such that $R^+ = A$.//

Theorem 3.5.2. If $(A,+)$ is an abelian group such that it is a direct sum $T \oplus C$ where T is a bounded group and C is a torsion free group admitting a Goldie ring structure with 1, then there exists a Goldie ring R such that $R^+ = A$.

Proof. Since T is a bounded group, we have an $m \in Z^+$ such that $mT = 0$, so by Theorem II 1.d. [6], T is of the type $T = \oplus\ C(P_j^{k_j})$, $k_j|m$. It follows from the proof of Theorem 3.5.1, that T can be given a Goldie ring structure with 1. Since C also admits a Goldie ring structure with 1, we then see that there exists a Goldie ring R such that $R^+ = A$.//

Theorem 3.5.3. The zero ring over any finitely generated additive group G is Goldie.

Proof. Let $G = \langle g_1, g_2, \ldots, g_n \rangle$ and R be the zero ring on G. Now by the converse of Proposition 126.1 [4], the zero ring on G is Noetherian and hence the ring R is Goldie by (page 8, [1]).//

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ON A SP-SASAKIAN MANIFOLD

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ABSTRACT : In the present paper we have studied an η-Einstein sp-Sasakian manifold. Some properties of curvature tensor of a semi symmetric non metric connection in an sp-Sasakian manifold have been also discussed.

1. INTRODUCTION

Let M^n be an n-dimensional C^∞-manifold. If there exists in M^n a tensor field ϕ of type (1, 1) consisting of a vector field ξ and 1-form η in M^n satisfying

$$\phi^2(X) = X - \eta(X)\xi, \qquad \phi(\xi) = 0, \qquad \eta(\xi) = 1. \qquad \ldots (1.1)$$

Then M^n is called an almost para contact manifold.

Let g be the Riemannian metric satisfying–

$$\eta(X) = g(X, \xi), \qquad \eta(\phi X) = 0, \qquad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \qquad \ldots (1.2)$$

Then the set (ϕ, ξ, η, g) satisfying (1.1) and (1.2) is called an almost para contact Riemannian structure. The manifold with such structure is called an almost para contact Riemannian manifold [1].

If we define $F(X, Y) = g(\phi, X, Y)$, then in addition to the above relation, the following are satisfied–

$$F(X, Y) = F(Y, X) \qquad \ldots (1.3)$$

$$F(\phi X, \phi Y) = F(X, Y) \qquad \ldots (1.4)$$

If in M^n the relation

$$(\nabla_X\eta)Y - (\nabla_Y\eta)(X) = 0 \qquad \ldots (1.5)$$

$(d\eta)(X, Y) = 0$ i.e. η is closed. ... (1.6)

$$(\nabla_X F)(Y, Z) = -\, g(X, Z)\eta(Y) - g(X, Y)\eta(Z) + 2\eta(X)\eta(Y)\eta(Z) \qquad \ldots (1.7)$$

$$(\nabla_X\eta)(Y) + (\nabla_Y\eta)(X) = 2F(X, Y) \qquad \ldots (1.8)$$

and $(\nabla_X\xi) = \phi(X)$ ... (1.9)

hold good then (M^n, g) is called para-Sasakian manifold or p-Sasakian manifold where ∇ denotes the Riemannian connection in M^n.

Further if in (M^n, g) the following relation also holds good

$$(\nabla_X \eta)(Y) = -g(X, Y) + \eta(X)\eta(Y) \qquad \ldots (1.10)$$

along with (1.1), (1.2) and (1.9), then it is called sp-Sasakian manifold [1].

In sp-Sasakian manifold M^n, the following relations also hold [2]

(a) $'R(X, Y, Z, \xi) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X)$ ... (1.11)

where $'R(X, Y, Z, W) = g(R(X, Y, Z), W)$

(b) $R(X, Y, \xi) = \eta(X)Y - \eta(Y)X$

(c) $S(X, \xi) = -(n - 1)\eta(X)$;

R and S are the curvature tensor and Ricci tensor respectively.

The D-conformal curvature tensor $B(X, Y, Z)$ in sp-Sasakian manifold M^n $(n>4)$ is defined by ([3], [4]) as follows

$$\begin{aligned} B(X, Y, Z) = {} & R(X, Y, Z) + \frac{1}{n-3}\,[S(X, Z)Y - S(Y, Z)X + g(X, Z)QY \\ & - g(Y, Z)QX - S(X, Z)\eta(Y)\xi + S(Y, Z)\eta(X)\xi - \eta(X)\eta(Z)QY + \eta(Y)\eta(Z)QX] \\ & - \frac{K-2}{n-3}\,[g(X, Z)Y - g(Y, Z)X] + \frac{K}{n-3}\,[g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi \\ & + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X] \qquad \ldots (1.12) \end{aligned}$$

where $K = \frac{r+2(n-1)}{n-2}$ and $S(X, Y) = g(QX, Y)$.

If the Ricci tensor $S(X, Y)$ satisfies the relation

$$S(X,Y) = \left(\frac{r}{n-1}+1\right)g(X,Y) - \left(\frac{r}{n-1}+n\right)\eta(X)\eta(Y) \qquad \ldots (1.13)$$

then the manifold M^n is called an η-Einstein sp-Sasakian manifold M^n, and the D-Conformal curvature tensor $B(X, Y, Z)$ reduceds to the tensor field $U(X, Y, Z)$ given by ([3] [4]).

$$\begin{aligned} U(X, Y, Z) = {} & R(X, Y, Z) + \frac{r+2(n-1)}{(n-1)(n-2)}\,[g(X, Z)Y - g(Y, Z)X] - \frac{r+n(n-1)}{(n-1)(n-2)} \\ & [g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X] \qquad \ldots (1.14) \end{aligned}$$

Finally in n-dimentional sp-Sasakian manifold we have [5]

$$(div\ B)(X,Y,Z) = \frac{n-4}{n-3}\tilde{B}(X,Y,Z), \qquad \text{... (1.15)}$$

where $\tilde{B}(X, Y, Z) = (\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) - \in[\eta(X)S(Y, Z) - \eta(Y)S(X, Z)]$

$$+ \left(\frac{r+2(n-1)}{n-2}\right) \in [\eta(X)g(Y, Z) - \eta(Y)g(X, Z)] - \frac{1}{2(n-2)}\ \{g(Y, Z) - \eta(Y)\eta(Z)\}dr(X)$$

$$- \{g(X, Z) - \eta(X)\eta(Z)\}\ dr(Y)]. \qquad \text{... (1.16)}$$

A semi-symmetric non-metric connection $\overline{\nabla}$ in an almost para contact metric manifold M^n can be defined as in [6]. It is given by

$$\overline{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X, Y)\xi - a(X)Y - a(Y)X \qquad \text{... (1.17)}$$

where η and a are 1–form associated with the vector field ξ and A on M^n given by–

$g(X, A) = a(X)$

$g(X, \xi) = \eta(X)$

2. η-EINSTEIN SP-SASAKIAN MANIFOLDS

Theorem 2.1. In an η-Einstein sp-Sasakian manifold,

$$'U(X, Y, Z, \xi) = 0 \qquad \text{... (2.1)}$$

$$'U(X, Y, Z, W) - {}'U(Z, W, X, Y) = 0 \qquad \text{... (2.2)}$$

$$'U(X, Y, Z, W) + {}'U(X, Y, W, Z) = 0 \qquad \text{... (2.3)}$$

$$U(X, Y, Z) + U(Y, Z, X) + U(Z, X, Y) = 0 \qquad \text{... (2.4)}$$

Proof. Form (1.11) and (1.14), we get (2.1). Similarly the other results can also be proved.

Theorem 2.2. An η-Einstein sp-Sasakian manifold satisfying the relation $R(X, Y, U) = 0$ is η-Einstein flat, where $R(X, Y)$ is considered as derivation of the tensor algebra at each point of the manifold for tangent vectors X, Y [7].

Proof. we have

$$(R(X, Y)U)(Z, W, V) = R(X, Y, U(Z, W, V)) - U(R(X, Y, Z), W, V)$$

$$- U(Z, R(X, Y, W), V) - U(Z, W, R(X, Y, V)). \qquad \text{... (2.5)}$$

Let us suppose that in Sasakian manifold.

$$R(X, Y, U) = 0. \qquad \text{... (2.6)}$$

Then from (2.5) and (2.6) we get

$$R(X, Y, U(Z, W, V)) - U(R(X, Y, Z), W, V) - U(Z, R(X, Y, W), V) - U(Z, W, R(X, Y, V)) = 0 \quad \dots (2.7)$$

Putting $X = \xi$ in (2.7) we get

$$R(\xi, Y, U(Z, W, V)) - U(R(\xi, Y, Z), W, V) - U(Z, R(\xi, Y, W), V) - U(Z, W, R(\xi, Y, V)) = 0 \quad \dots (2.8)$$

In view of (1.11), (2.1) and (2.5), (2.8) reduces to

$$'U(Z, W, V, Y)\xi - \eta(U(Z, W, V))Y - g(Y, Z)U(W, \xi, V) + \eta(Z)U(Y, W, V) - g(Y, W)U(Z, \xi, V) + \eta(W)U(Z, Y, V) - g(Y, V)U(Z, W, \xi) + \eta(V)U(Z, W, Y) = 0 \quad \dots (2.9)$$

Using (2.1), (2.3) in (2.9) we get,

$$'U(Z, W, V, Y)\xi + \eta(Z)U(Y, W, V) + \eta(W)U(Z, Y, V) + \eta(V)U(Z, W, Y) = 0 \quad \dots (2.10)$$

where $'U(Z, W, V, Y) = g(U(Z, W, V)Y)$.

Again putting $V = \xi$ in (2.10) and then using (1.1), (2.1), we get

$U(Z, W, Y) = 0$ for all vector field Z, W, Y.

3. SP-SASAKIAN MANIFOLD WITH DIV B.

Theorem 3.1. In an sp-Sasakian manifold

$$(\text{div } B)(X, Y, \xi) = 0 \quad \dots (3.1)$$

$$(\text{div } B)(X, Y, Z) + (\text{div } B)(Y, X, Z) = 0 \quad \dots (3.2)$$

$$(\text{div } B)(X, Y, Z) + (\text{div } B)(Y, Z, X) + (\text{div } B)(Z, X, Y) = 0 \quad \dots (3.3)$$

$$(\text{div } B)(X, Y, Z) + (\text{div } B)(Z, X, Y) - (\text{div } B)(Z, Y, X) = 0 \quad \dots (3.4)$$

Proof. put ξ for Z in (1.15) we get

$$(\text{div } B)(X, Y, \xi) = \left(\frac{n-4}{n-3}\right)\tilde{B}(X, Y, \xi)$$

which gives on simplification, we have

$$(\text{div } B)(X, Y, \xi) = \left(\frac{n-4}{n-3}\right) [(\nabla_X S)(Y, \xi) - (\nabla_Y S)(X, \xi)] \quad \dots (3.5)$$

Now $(\nabla_X S)(Y, \xi) = \nabla_X(S(Y, \xi)) - S(\nabla_X Y, \xi) - S(Y, \nabla_X \xi)$ ... (3.6)

Using (1.9) and (1.11) c. in (3.6), we get

$(\nabla_X S)(Y, \xi) = -(n-1)(\nabla_X \eta)(Y) - S(Y, \phi X).$ ... (3.7)

By virtue of $\phi(X, Y) = (\nabla_X \eta)(Y)$, equation (3.7) can be put as

$(\nabla_X S)(Y, \xi) = -(n-1)\phi(X, Y) - S(\phi X, Y)$ ... (3.8)

From (3.8) and (3.5), we get –

$$(\text{div } B)(X, Y, \xi) = \left(\frac{n-4}{n-3}\right)[S(X, \phi Y) - S(\phi X, Y)]$$

i.e. $(\text{div } B)(X, Y, \xi) = 0$ which proves (3.1).

With the help of (1.16) we can obtain (3.2), (3.3) and (3.4) respectively.

4. CURVATURE TENSOR

Let $\bar{R}$ and R be the curvature tensor of $\overline{\nabla}$ and ∇ in an almost para contact metric manifold. In view of (1.17) we have from [6].

$$\bar{R}(X, Y, Z) = R(X, Y, Z) - \alpha(Y, Z)X + \alpha(X, Z)Y - g(Y, Z)QX + g(X, Z)QY$$
$$- 2da(X, Y)Z + \beta(Y, Z)X - \beta(X, Z)Y. \quad \text{... (4.1)}$$

where $\alpha(Y, Z) = g(QY, Z) = (\nabla_Y \eta)(Z) - \eta(Y)\eta(Z) + \frac{1}{2}\eta(U)g(Y, Z).$

$QX = \nabla_X U - \eta(X)U + \frac{1}{2}\eta(U)X$

and $\beta(Y, Z) = (\nabla_Y a)(Z) - a(Y)\eta(Z) + a(Y)a(Z) - \eta(Y)a(Z) + a(U)g(Y, Z).$

From (1.10) and (4.1), we have

$$'\bar{R}(X, Y, Z, W) = {}'R(X, Y, Z, W) + g(Y, Z)g(X, W) - g(X, Z)g(Y, W)$$
$$- a(Y)\eta(Z)g(X, W) - \eta(Y)a(Z)g(X, W) + a(X)\eta(Z)g(Y, W) + \eta(X)a(Z)g(Y, W) \quad \text{... (4.2)}$$

where $'\bar{R}(X, Y, Z, W) = g(\bar{R}(X, Y, Z), W).$

Contracting (4.1) w.r.t. X we get

$\bar{S}(Y, Z) = S(Y, Z) + (n-1)g(Y, Z) - (n-1)a(Y)\eta(Z) - (n-1)\eta(Y)a(Z),$ ... (4.3)

where $\bar{S}$ and S are the Ricci tensor $\overline{\nabla}$ and ∇ respectively.

Also $\bar{r} = r + (n-1)(n-2),$ ... (4.4)

where $\bar{r}$ and r are scalar curvature of the connection $\overline{\nabla}$ and ∇ respectively.

Theorem 4.1. If in an sp-Sasakian manifold the curvature tensor of a semi-symmetric non-metric connection vanishes, then the manifold is projectively flat.

Proof. Since $\overline{R}(X, Y, Z) = 0$, So (4.2) gives

$$R(X, Y, Z) = g(X, Z)Y - g(Y, Z)X + a(Y)\eta(Z)X + \eta(Y)a(Z)X - a(X)\eta(Z)Y - \eta(X)a(Z)Y. \qquad \text{... (4.5)}$$

On contraction, we get

$$S(Y, Z) = -(n-1)g(Y, Z) + (n-1)a(Y)\eta(Z) + (n-1)\eta(Y)a(Z). \qquad \text{... (4.6)}$$

From (4.5) and (4.6) we get

$$R(X, Y, Z) - \frac{1}{n-1}\,[S(Y, Z)X - S(X, Z)Y] = 0$$

or, $P = 0$,

where P is the projective curvature tensor of the manifold M^n [7] defined as

$$P(X, Y, Z) = R(X, Y, Z) - \frac{1}{n-1}\,[S(Y, Z)X - S(X, Z)Y].$$

Theorem 4.2. If in an sp-Sasakian manifold, the Ricci tensor of a semi-symmetric non-metric connection $\overline{\nabla}$ vanishes, then the curvature tensor $\overline{\nabla}$ is equal to the projective curvature tensor of the manifold.

Proof. Since $\overline{S}(Y, Z) = 0$ then equation (4.3) gives,

$$g(Y, Z) = -\frac{1}{n-1}S(Y, Z) + a(Y)\eta(Z) + \eta(Y)a(Z). \qquad \text{... (4.7)}$$

From (4.2) and (4.7), we have-

$$\overline{R}(X, Y, Z) = P(X, Y, Z),$$

where P is the projective curvature tensor of the manifold.

Theorem 4.3. In an sp-Sasakian manifold, the projective curvature tensor of a semi-symmetric non-metric connection $\overline{\nabla}$ is equal to the projective curvature tensor ∇ of the manifold.

Proof. From (4.2) and (4.3) we get

$$\overline{R}(X, Y, Z) = R(X, Y, Z) + \frac{1}{n-1}[\overline{S}(Y, Z) - S(Y, Z)]X - \frac{1}{n-1}[\overline{S}(X, Z) - S(X, Z)]Y$$

or, $\overline{R}(X, Y, Z) - \frac{1}{n-1}[\overline{S}(Y, Z)X - \overline{S}(X, Z)Y] = R(X, Y, Z) - \frac{1}{n-1}[S(Y, Z)X - S(X, Z)]Y$

or, $\overline{P}(X, Y, Z) = P(X, Y, Z)$,

where P is the projective curvature tensor of the manifold.

Theorem 4.4 An sp-Sasakian manifold with semi-symmetric non-metric connection $\overline{\nabla}$ gives the following identities.

$$\bar{R}(X, Y, Z) + \bar{R}(Y, Z, X) + \bar{R}(Z, X, Y) = 0 \qquad \ldots (4.8)\ (a)$$

$$\bar{R}(X, Y, Z) + \bar{R}(Y, X, Z) = 0 \qquad \ldots (4.8)\ (b)$$

Proof. Using Bianchi's first identity (4.2) gives (4.8)a, and (4.8)b, respectively.

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MULTIDIMENSIONAL LAGUERRE TRANSFORMS

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ABSTRACT : In the present paper, we introduce multidimensional Laguerre transform to present its certain interesting applications to the theory of generalized multiple hypergeometric functions of several variables including multivariable *H*-function of Srivastava and Panda [11, 12]. The various operational formulas thus obtained are believed to be new, these results may be useful in deriving new and known properties of special functions involved. In the last section, we shall also introduce another multidimensional Laguerre transform to derive interesting results.

1. INTRODUCTION

Chandel [1] introduced multidimensional Laplacian oprator to give integral representations of Lauricella's multiple hypergeometric functions of several variables [9]. Chandel [2] further used this operator to give integral representations of multiple hypergeometric functions ${}^{(k)}_{(1)}E^{(n)}_{D}$ and ${}^{(k)}_{(2)}E^{(n)}_{D}$ of Exton [7, 8]. Further Chandel and Dwivedi [3, 4] introduced multidimensional Whittaker transforms of Lauricella's multiple hypergeometric functions [9], Exton [7, 8] and generalized multiple hypergeometric function of Srivastava and Daoust [10] (also see Srivastava and Manocha [14, p. 64 (18), (19), (20)]). Recently Chandel and Kumar [5] have made applications of above multidimensional integral transforms to derive the results involving Srivastava and Panda's *H*-function of several complex variables [11, 12].

In the present paper, making an appeal to the result due to Erdélyi et. al. [6, p. 292]

$$\int_0^\infty Z^{\rho} e^{-z} L_n^{(\alpha)}(z)dz = \frac{(-1)^n\Gamma(\rho-\alpha+1)\Gamma(\rho+1)}{n!\Gamma(\rho-\alpha-n+1)},\ Re\ (\rho) > -1$$

We introduce multidimensional Laguerre transform defined by

$$(1.1)\quad L^{(\alpha,\beta,m)}_{\gamma_1\dots\gamma_n}\{\} = \frac{(-1)^m m!\Gamma(\beta-\alpha-m+\gamma_1+\dots+\gamma_n)\Gamma(\gamma_1+\dots+\gamma_n)}{\Gamma(\beta-\alpha+\gamma_1+\dots+\gamma_n)\Gamma(\beta+\gamma_1+\dots+\gamma_n)\Gamma(\gamma_1)\dots\Gamma(\gamma_n)}$$

$$\int_0^\infty \dots \int_0^\infty e^{-(x_1+\dots+x_n)}(x_1+\dots+x_n)^\beta x_1^{\gamma_1-1}\dots x_n^{\gamma_n-1} L_m^{(\alpha)}(x_1+\dots+x_n)\{\}dx_1\dots dx_n,$$

where $Re(\beta-\alpha-m+\gamma_1+\dots+\gamma_n)>0$; m, n are arbitrary positive integers $Re(\gamma_j)>0$, $j=1,\ \dots,\ n$,

and present its certain interesting applications to the theory of generalized multiple hypergeometric functions of several variables including multivariable H-function of Srivastava and Panda ([11], [12]; also see [13], p. 251). The various operational formulas thus obtained are belived to be new and may be useful in deriving new and known properties of special functions involved.

In the last section, we shall also introduce another generalized multidimensional Laguerre transform to derive some interesting results.

2. APPLICATIONS OF MULTIDIMENSIONAL LAGUERRE TRANSFROM.

In this section, as applications of Laguerre transform, we obtain the following results:

$$(2.1)\quad L_{\gamma_1\dots\gamma_n}^{(\alpha,\beta,m)}\{1\}=1,$$

$$(2.2)\quad L_{\gamma_1\dots\gamma_n}^{(\alpha,\beta,m)}\{x_1^{m_1}\dots x_n^{m_n}\}=$$

$$\frac{(\beta-\alpha+\gamma_1+\dots+\gamma_n)_{m_1+\dots+m_n}(\beta+\gamma_1+\dots+\gamma_n)m_1+\dots+m_n(\gamma_1)_{m_1}\dots(\gamma_n)_{m_n}}{(\beta-\alpha-m+\gamma_1+\dots+\gamma_n)_{m_1+\dots+m_n}(\gamma_1+\dots+\gamma_n)_{m_1+\dots+m_n}}$$

$$(2.3)\quad L_{\gamma_1\dots\gamma_n}^{(\alpha,\beta,m)}\{{}_1F_1(\beta-\alpha-m+\gamma_1+\dots+\gamma_n;\beta-\alpha+\gamma_1+\dots+\gamma_n;\ u_1x_1+\dots+u_nx_n)\}$$

$$=F_D^{(n)}(\beta+\gamma_1+\dots+\gamma_n,\ \gamma_1+\dots+\gamma_n;\ \gamma_1,\dots,\gamma_n;\ u_1,\dots,u_n),$$

$$(2.4)\quad L_{\gamma_1\dots\gamma_n}^{(\alpha,\beta,m)}\{{}_1F_1(\gamma_1+\dots+\gamma_n;\ \beta-\alpha+\gamma_1+\dots+\gamma_n;\ u_1x_1+\dots+u_nx_n)\}$$

$$=F_D^{(n)}(\beta+\gamma_1+\dots+\gamma_n,\gamma_1\dots\gamma_n;\ \beta-\alpha-m+\gamma_1+\dots+\gamma_n;u_1\dots u_n),$$

$$(2.5)\quad L_{\gamma_1\dots\gamma_n}^{(\alpha,\beta,m)}\{{}_1F_1(\gamma_1+\dots+\gamma_n;\ \beta+\gamma_1+\dots+\gamma_n;\ u_1x_1+\dots+u_nx_n)\}$$

$$=F_D^{(n)}(\gamma_1+\dots+\gamma_n+\beta-\alpha,\ \gamma_1\dots\gamma_n;\ \beta-\alpha-m+\gamma_1+\dots+\gamma_n;\ u_1\dots u_n),$$

$$(2.6)\quad L^{(\alpha,\beta,m)}_{\gamma_1\ldots\gamma_n}\{{}_1F_1(\beta-\alpha-m+\gamma_1+\ldots+\gamma_n;\ \beta+\gamma_1+\ldots+\gamma_n;\ u_1x_1+\ldots+u_nx_n)\}$$

$$= F^{(n)}_D(\beta-\alpha+\gamma_1+\ldots+\gamma_n,\ \gamma_1\ldots\gamma_n;\ \gamma_1+\ldots\gamma_n;\ u_1\ldots u_n)$$

$$(2.7)\quad L^{(\alpha,\beta,m)}_{\gamma_1\ldots\gamma_n}\left\{(x_1+\ldots+x_n)^{m_1+\ldots+m_n}\right\}=\frac{(\beta-\alpha+\gamma_1+\ldots+\gamma_n)_{m_1+\ldots+m_n}(\beta+\gamma_1+\ldots+\gamma_n)_{m_1+\ldots+m_n}}{(\beta-\alpha-m+\gamma_1+\ldots+\gamma_n)_{m_1+\ldots+m_n}}$$

$$(2.8)\quad L^{(\alpha,\beta,m)}_{\gamma_1\ldots\gamma_n}\left\{\phi_2^{(n)}\left(\beta_1,\ldots,\beta_n;\beta-\alpha+\gamma_1+\ldots+\gamma_n;u_1(x_1+\ldots+x_n),\ldots,u_n\left(x_1+\ldots+x_n\right)\right)\right\}$$

$$= F^{(n)}_D(\beta+\gamma_1+\ldots+\gamma_n,\ \beta_1,\ldots,\ \beta_n;\ \beta-\alpha-m+\gamma_1+\ldots+\gamma_n;\ u_1,\ldots,u_n),$$

$$(2.9)\quad L^{(\alpha,\beta,m)}_{\gamma_1\ldots\gamma_n}\left\{\phi_2^{(n)}\left(\beta_1,\ldots,\beta_n;\beta+\gamma_1+\ldots+\gamma_n;u_1(x_1+\ldots+x_n),\ldots,u_n\left(x_1+\ldots+x_n\right)\right)\right\}$$

$$= F^{(n)}_D(\beta-\alpha+\gamma_1+\ldots+\gamma_n,\ \beta_1,\ldots,\ \beta_n;\ \beta-\alpha-m+\gamma_1+\ldots+\gamma_n;\ u_1,\ldots,u_n),$$

$$(2.10)\quad L^{(\alpha,\beta,m)}_{\gamma_1,\ldots\gamma_n}\left\{\psi_2^{(n)}(\beta-\alpha-m+\gamma_1+\ldots+\gamma_n;\beta_1,\ldots,\beta_n;u_1(x_1+\ldots+x_n),\ldots,U_n(x_1+\ldots+x_n))\right\}$$

$$= F^{(n)}_C(\beta-\alpha+\gamma_1+\ldots\gamma_n,\ \beta+\gamma_1+\ldots+\gamma_n;\ \beta_1,\ldots,\beta_n;\ u_1,\ldots u_n),$$

$$(2.11)\quad L^{(\alpha,\beta,m)}_{\gamma_1\ldots\gamma_n}\left\{\exp\left[\sum_{i=1}^{n}u_i(x_1+\ldots+x_n)\right]\right\}$$

$$= {}_2\mathrm{F}_1[\beta-\alpha+\gamma_1+\ldots+\gamma_n,\ \beta+\gamma_1+\ldots+\gamma_n;\ \beta-\alpha-m+\gamma_1+\ldots+\gamma_n;\ u_1+\ldots+u_n],$$

3. SPECIAL CASES

Case (a) For $\beta=\alpha$, we derive

$$(3.1)\quad L^{(\alpha,\alpha,m)}_{\gamma_1\ldots\gamma_n}\left\{x_1^{m_1}\ldots x_n^{m_n}\right\}=\frac{(\alpha+\gamma_1+\ldots+\gamma_n)_{m_1+\ldots+m_n}(\gamma_1)_{m_1}\cdots(\gamma_n)_{m_n}}{(\gamma_1+\ldots+\gamma_n-m)_{m_1+\ldots+m_2}}$$

$$(3.2)\quad L^{(\alpha,\alpha,m)}_{\gamma_1\ldots\gamma_n}\{\exp(u_1x_1+\ldots+u_nx_n)\}$$

$$= F_D^{(n)}(\alpha + \gamma_1 + \ldots + \gamma_n, \gamma_1,\ldots,\gamma_n;\ \gamma_1 + \ldots\ \gamma_n - m;\ u_1,\ldots,u_n),$$

(3.3) $$L_{\gamma_1\ldots\gamma_n}^{(\alpha,\alpha,m)}\left\{\psi_2^{(n)}(\gamma_1+\ldots+\gamma_n - m;\beta_1,\ldots,\beta_n;u_1x_1,\ldots,u_nx_n)\right\}$$

$$= F_A^{(n)}(\alpha + \gamma_1 + \ldots + \gamma_n, \gamma_1,\ldots,\gamma_n;\ \beta_1,\ldots,\beta_n;\ u_1,\ldots,u_n),$$

(3.4) $$L_{\gamma_1\ldots\gamma_n}^{(\alpha,\alpha,m)}\left\{\psi_2^{(n)}(a;\gamma_1,\ldots,\gamma_n;u_1x_1,\ldots,u_nx_n)\right\}$$

$$= {}_2F_1(a,\ \alpha + \gamma_1 + \ldots + \gamma_n;\ \gamma_1 + \ldots + \gamma_n - m;\ u_1 + \ldots + u_n),$$

(3.5) $$L_{\gamma_1\ldots\gamma_n}^{(\alpha,\alpha,m)}\left\{\psi_2^{(n)}(\gamma_1+\ldots+\gamma_n - m;\gamma_1,\ldots,\gamma_n;u_1x_1,\ldots,u_nx_n)\right\}$$

$$= \left[1-(u_1+\ldots+u_n)\right]^{m-(\gamma_1+\ldots+\gamma_n)}$$

(3.6) $$L_{\gamma_1\ldots\gamma_n}^{(\alpha,\alpha,m)}\left\{\phi_2^{(n)}(\beta_1,\ldots,\beta_n;\alpha + \gamma_1+\ldots+\gamma_n;u_1x_1,\ldots,u_nx_n)\right\}$$

$$= F_B^{(n)}(\beta_1,\ldots,\beta_n, \gamma_1,\ldots,\gamma_n;\ \gamma_1 + \ldots + \gamma_n - m;\ u_1,\ldots,u_n),$$

(3.7) $$L_{\gamma_1\ldots\gamma_n}^{(\alpha,\alpha,m)}\{\psi_2(\gamma_1+\ldots+\gamma_n - m;\beta,\beta';u_1x_1+\ldots u_kx_k, u_{k+1}x_{k+1}+\ldots+u_nx_n)\}$$

$$= {}^{(k)}_{(1)}E_D^{(n)}(\alpha + \gamma_1 + \ldots + \gamma_n, \gamma_1,\ldots\gamma_n;\ \beta,\ \beta';\ u_1,\ldots,u_n),$$

(3.8) $$L_{\gamma_1\ldots\gamma_n}^{(\alpha,\alpha,m)}\{\phi_2(\beta,\beta';\alpha + \gamma_1+\ldots+\gamma_n;u_1x_1+\ldots+u_kx_k, u_{k+1}x_{k+1}+\ldots+u_nx_n)\}$$

$$= {}^{(k)}_{(2)}E_D^{(n)}(\beta,\beta';\ \gamma_1,\ldots,\gamma_n;\ \gamma_1+\ldots+\gamma_n - m;\ u_1,\ldots,u_n),$$

(3.9) $$L_{\gamma_1\ldots\gamma_n}^{(\alpha,\alpha,m)}\left\{\cdot\prod_{i=1}^{n} L_{n_i}^{(\gamma_i-1)}(u_ix_i)\right\}$$

$$= \prod_{i=1}^{n}\frac{(n_i)!}{(\gamma_i)_{n_i}} F_D^{(n)}(\alpha+\gamma_1+\ldots+\gamma_n, -n_1,\ldots,-n_n;\gamma_1+\ldots+\gamma_n - m;u_1,\ldots,u_n)\ .$$

and

$$L^{(\alpha,\alpha,m)}_{\gamma_1\cdots\gamma_n}\left\{{}^{(k)}_{(1)}\phi^{(n)}_{CD}(\gamma_1+\ldots+\gamma_n-m;b;c,\gamma_{k+1},\ldots,\gamma_n;u_1x_1,\ldots,u_nx_n)\right\} \tag{3.10}$$

$$={}^{(k)}_{(4)}\phi^{(n)}_{CD}(\alpha+\gamma_1+\ldots+\gamma_n,b,\gamma_1,\ldots,\gamma_k;c;u_1,\ldots,u_n)$$

Case (b) For $\beta = 0$, we derive

$$L^{(\alpha,0,m)}_{\gamma_1\cdots\gamma_n}\left\{x_1^{m_1}\ldots x_n^{m_n}\right\}=\frac{(\gamma_1+\ldots+\gamma_n-\alpha)_{m_1+\ldots m_n}(\gamma_1)_{m_1}\cdots(\gamma_n)_{m_n}}{(\gamma_1+\ldots+\gamma_n-\alpha-m)_{m_1+\ldots+m_n}}, \tag{3.11}$$

$$L^{(\alpha,0,m)}_{\gamma_1\cdots\gamma_n}\left\{\phi_2^{(n)}(\beta_1,\ldots,\beta_n;\gamma_1+\ldots+\gamma_n-\alpha;u_1x_1,\ldots,u_nx_n)\right\} \tag{3.12}$$

$$=F_B^{(n)}(\beta_1,\ldots,\beta_n,\gamma_1,\ldots\gamma_n;\gamma_1+\ldots+\gamma_n-\alpha-m;u_1,\ldots u_n),$$

$$L^{(\alpha,0,m)}_{\gamma_1\cdots\gamma_n}\left\{\psi_2^{(n)}(\gamma_1+\ldots+\gamma_n-\alpha-m;\beta_1,\ldots,\beta_n;u_1x_1,\ldots,u_nx_n\right\} \tag{3.13}$$

$$=F_A^{(n)}(\gamma_1,\ldots\gamma_n-\alpha,\gamma_1,\ldots\gamma_n;\beta_1,\ldots,\beta_n;u_1,\ldots u_n),$$

$$L^{(\alpha,0,m)}_{\gamma_1\cdots\gamma_n}\left\{\psi_2^{(n)}(\gamma_1+\ldots+\gamma_n-\alpha-m;\gamma_1,\ldots,\gamma_n;u_1x_1,\ldots,u_nx_n\right\} \tag{3.14}$$

$$=\left[1-(u_1+\ldots+u_n)\right]^{-(\gamma_1+\ldots+\gamma_n-\alpha)},$$

$$L^{(\alpha,0,m)}_{\gamma_1\cdots\gamma_n}\left\{{}_0F_1(-;\gamma_1+\ldots+\gamma_n-\alpha;u_1x_1+\ldots+u_nx_n\right\} \tag{3.15}$$

$$=\phi_2^{(n)}(\gamma_1,\ldots,\gamma_n;\gamma_1+\ldots+\gamma_n-\alpha-m;u_1,\ldots,u_n),$$

$$L^{(\alpha,0,m)}_{\gamma_1\cdots\gamma_n}\left\{\exp(u_1x_1+\ldots+u_nx_n)\right\} \tag{3.16}$$

$$=F_D^{(n)}(\gamma_1+\ldots+\gamma_n-\alpha,\gamma_1,\ldots;\gamma_n;\gamma_1+\ldots+\gamma_n-\alpha-m;u_1,\ldots,u_n),$$

$$L^{(\alpha,0,m)}_{\gamma_1\cdots\gamma_n}\left\{\psi_2(\gamma_1+\ldots+\gamma_n-\alpha-m;\beta,\beta';u_1x_1+\ldots+u_kx_k,u_{k+1}x_{k+1}+\ldots+u_nx_n)\right\} \tag{3.17}$$

$$\overset{(k)}{\underset{(1)}{=}} E_D^{(n)}(\gamma_1+\ldots+\gamma_n-\alpha,\gamma_1,\ldots,\gamma_n;\beta,\beta';u_1,\ldots,u_n),$$

$$(3.18)\quad L^{(\alpha,0,m)}_{\gamma_1\ldots\gamma_n}\{\phi_2(\beta,\beta';\gamma_1+\ldots+\gamma_n-\alpha;u_1x_1+\ldots+u_kx_k,u_{k+1}x_{k+1}+\ldots+u_nx_n)\}$$

$$\overset{(k)}{\underset{(2)}{=}} E_D^{(n)}(\beta,\beta';\gamma_1,\ldots\gamma_n;\gamma_1+\ldots+\gamma_n-\alpha-m,;u_1,\ldots,u_n),$$

$$(3.19)\quad L^{(\alpha,0,m)}_{\gamma_1\ldots\gamma_n}\left\{\prod_{i=1}^{n}L_{n_i}^{(\gamma_i-1)}(u_ix_i)\right\}$$

$$\prod_{i=1}^{n}\frac{(n_i)!}{(\gamma_i)_{n_i}}F_D^{(n)}(\gamma_1+\ldots+\gamma_n-\alpha,-n_1,\ldots,-n_n;\gamma_1+\ldots+\gamma_n-\alpha-m;u_1,\ldots u_n).$$

4. APPLICATION TO MULTIPLE HYPERGEOMETRIC FUNCTION OF SRIVASTAVA AND DAOUST

In this section, we derive the following multidimensional Laguerre transforms involving multiple hypergeometric function of Srivastava and Daoust [10, 14] :

$$(4.1)\quad L^{(\alpha,\beta,m)}_{\gamma_1\ldots\gamma_n}\left\{F^{A:B';\ldots B^{(n)}}_{C:D';\ldots;D^{(n)}}\left(\begin{array}{l}\left[(a):\theta',\ldots,\theta^{(n)}\right]:\left[(b'):\phi'\right];\ldots;\left[\left(b^{(n)}\right):\phi^{(n)}\right];\\ \left[(c):\psi',\ldots,\psi^{(n)}\right]:\left[(d'):\delta'\right];\ldots;\left[\left(d^{(n)}\right):\delta^{(n)}\right];\end{array}\right.\right.$$

$$\left.\left.u_1x_1^{\xi_1},\ldots,u_nx_n^{\xi_n}\right)\right\}=F^{A+2:B'+1;\ldots B^{(n)}+1}_{C+2:D';\ldots;D^{(n)}}\left(\begin{array}{l}\left[(a):\theta',\ldots,\theta^{(n)}\right],\\ \left[(c):\psi',\ldots,\psi^{(n)}\right],\end{array}\right.$$

$$\begin{array}{l}\left[\beta-\alpha+\gamma_1+\ldots+\gamma_n:\xi_1,\ldots,\xi_n\right],\left[\beta+\gamma_1+\ldots+\gamma_n:\xi_1,\ldots,\xi_n\right]:\\ \left[\beta-\alpha-m+\gamma_1+\ldots+\gamma_n:\xi_1,\ldots,\xi_n\right],\left[\gamma_1+\ldots+\gamma_n:\xi_1,\ldots,\xi_n\right]:\end{array}$$

$$\left.\begin{array}{ccc}\left[(b'):\phi'\right],\left[\gamma_1:\xi_1\right];\ldots; & \left[\left(b^{(n)}\right):\phi^{(n)}\right],\left[\gamma_n:\xi_n\right]; & \\ \left[(d'):\delta'\right]\quad;\ldots; & \left[\left(d^{(n)}\right):\delta^{(n)}\right]; & u_1,\ldots,u_n\end{array}\right)$$

where $Re(\beta - \alpha - m + \gamma_1 + \ldots + \gamma_n) > 0$, m, n are any positive integers,

$$\mathrm{Re}(\gamma_1) > 0, 1 - \xi_i + \sum_{j=1}^{C} \psi_j^{(i)} + \sum_{j=1}^{D^{(i)}} \delta_j^{(i)} - \sum_{j=1}^{A} \theta_j^{(i)} - \sum_{j=1}^{B^{(i)}} \phi_j^{(i)} > 0; i = 1, \ldots, n,$$

$$(4.2) \quad L_{\gamma_1 \ldots \gamma_n}^{(\alpha,\beta,m)} \left\{ F_{C:D';\ldots;D^{(n)}}^{A:B';\ldots B^{(n)}} \left(\begin{matrix} \left[(a):\theta', \ldots, \theta^{(n)}\right]:\left[(b'):\phi'\right];\ldots;\left[\left(b^{(n)}\right):\phi^{(n)}\right]; \\ \left[(c):\psi', \ldots, \psi^{(n)}\right]:\left[(d'):\delta'\right];\ldots;\left[\left(d^{(n)}\right):\delta^{(n)}\right]; \end{matrix} \right. \right.$$

$$\left. \left. u_1 (x_1 + \ldots + x_n)^{\eta_1}, \ldots, u_n (x_1 + \ldots + x_n)^{\eta_n} \right) \right\}$$

$$= F_{C+1:D';\ldots;D^{(n)}}^{A+2:B';\ldots;B^{(n)}} \left(\begin{matrix} \left[(a):\theta', \ldots, \theta^{(n)}\right], \\ \left[(c):\psi', \ldots, \psi^{(n)}\right], \end{matrix} \right.$$

$$\begin{matrix} \left[\beta - \alpha + \gamma_1 + \ldots + \gamma_n : \eta_1, \ldots, \eta_n\right], \quad \left[\beta + \gamma_1 + \ldots + \gamma_n : \eta_1, \ldots, \eta_n\right]; \\ \left[\beta - \alpha - m + \gamma_1 + \ldots + \gamma_n : \eta_1, \ldots, \eta_n\right]: \end{matrix}$$

$$\left. \begin{matrix} \left[(b'):\phi'\right];\ldots;\left[\left(b^{(n)}\right):\phi^{(n)}\right]; \\ \left[(d'):\delta'\right];\ldots;\left[\left(d^{(n)}\right):\delta^{(n)}\right]; \end{matrix} u_1, \ldots, u_n \right),$$

where $Re(\beta - \alpha - m + \gamma_1 + \ldots + \gamma_n) > 0$, m, n are any positive integers,

$$\mathrm{Re}(\gamma_i) > 0, 1 - \eta_i + \sum_{j=1}^{C} \psi_j^{(i)} + \sum_{j=1}^{D^{(i)}} \delta_j^{(i)} - \sum_{j=1}^{A} \theta_j^{(i)} - \sum_{j=1}^{B^{(i)}} \phi_j^{(i)} > 0; \; i = 1, \ldots, n,$$

$$(4.3) \quad L_{\gamma_1,\ldots,\gamma_n}^{(\alpha,\beta,m)} \left\{ x_1^{m_1 \xi_1} \ldots x_n^{m_n \xi_n} (x_1 + \ldots + x_n)^{m_1 \eta_1 + \ldots + m_n \eta_n} \right\}$$

$$= \frac{(\beta + \gamma_1 + \ldots + \gamma_n - \alpha)_{m_1(\xi_1 + \eta_1) + \ldots + m_n(\xi_n + \eta_n)} (\beta + \gamma_1 + \ldots + \gamma_n)_{m_1(\xi_1 + \eta_1) + \ldots + m_n(\xi_n + \eta_n)} \, (\gamma_1)_{m_1 \xi_1} \cdots (\gamma_n)_{m_n \xi_n}}{(\beta + \gamma_1 + \ldots + \gamma_n - \alpha - m)_{m_1(\xi_1 + \eta_1) + \ldots + m_n(\xi_n + \eta_n)} (\gamma_1 + \ldots + \gamma_n)_{m_1 \xi_1 + \ldots + m_n \xi_n}}$$

and

$$(4.4)\quad L_{\gamma_1,\ldots,\gamma_n}^{(\alpha,\beta,m)}\left\{F_{C:D';\ldots;D^{(n)}}^{A:B';\ldots;B^{(n)}}\left(\begin{matrix}\left[(a):\theta',\ldots,\theta^{(n)}\right]:[(b'):\phi'];\ldots;\left[\left(b^{(n)}\right):\phi^{(n)}\right];\\ \left[(c):\psi',\ldots,\psi^{(n)}\right]:[(d'):\delta'];\ldots;\left[\left(d^{(n)}\right):\delta^{(n)}\right];\end{matrix}\right.\right.$$

$$\left.\left.u_1x_1^{\xi_1}(x_1+\ldots+x_n)^{\eta_1},\ldots,u_nx_n^{\xi_n}(x_1+\ldots+x_n)^{\eta_n}\right)\right\}$$

$$=F_{C+2:D';\ldots;D^{(n)}}^{A+2:B'+1;\ldots;B^{(n)}+1}\left(\begin{matrix}\left[(a):\theta',\ldots,\theta^{(n)}\right], & [\beta+\gamma_1+\ldots+\gamma_n-\alpha:\xi_1+\eta_1,\ldots,\xi_n+\eta_n],\\ \left[(c):\psi',\ldots,\psi^{(n)}\right], & [\beta+\gamma_1+\ldots+\gamma_n-\alpha-m:\xi_1+\eta_1,\ldots,\xi_n+\eta_n],\end{matrix}\right.$$

$$\begin{matrix}[\beta+\gamma_1+\ldots+\gamma_n:\xi_1+\eta_1,\ldots,\xi_n+\eta_n]:[(b'):\phi'],[\gamma_1:\xi_1];\ldots;\left[\left(b^{(n)}\right):\phi^{(n)}\right],[\gamma_n:\xi_n];\\ [\gamma_1+\ldots+\gamma_n:\xi_1,\ldots,\xi_n]: \qquad [(d'):\delta'] \qquad ;\ldots; \qquad \left[\left(d^{(n)}\right):\delta^{(n)}\right];\end{matrix}$$

$$\left.u_1,\ldots u_n\right),$$

provided that $Re(\beta-\alpha-m+\gamma_1+\ldots+\gamma_n)>0$, m, n are any positive integers,

$Re(\gamma_i)>0$ and $1-(\xi_i+\eta_i)+\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\phi_j^{(i)}>0$; $i=1,\ldots,n$.

5. APPLICATION TO MULTIVARIABLE *H*-FUNCTION OF SRIVASTAVA AND PANDA.

In this section, we derive following multidimensional Laguerre transform involving multivariable *H*-function of Srivastava and Panda [11, 12].

$$(5.1)\quad L_{\gamma_1\ldots\gamma_n}^{(\alpha,\beta,m)}\left\{H_{A,C:[B',D'];\ldots;\left[B^{(n)},D^{(n)}\right]}^{0,\gamma:(\mu',\nu');\ldots;\left(\mu^{(n)},\nu^{(n)}\right)}\left(\begin{matrix}\left[(a):\theta',\ldots,\theta^{(n)}\right]:[(b'):\phi'];\ldots;\left[\left(b^{(n)}\right):\phi^{(n)}\right];\\ \left[(c):\psi',\ldots,\psi^{(n)}\right]:[(d'):\delta'];\ldots;\left[\left(d^{(n)}\right):\delta^{(n)}\right];\end{matrix}\right.\right.$$

$$\left.\left.u_1x_1^{\xi_1}(x_1+\ldots+x_n)^{\eta_1},\ldots u_nx_n^{\xi_n}(x_1+\ldots+x_n)^{\eta_n}\right)\right\}$$

$$=\frac{\Gamma(\beta-\alpha-m+\gamma_1+\ldots+\gamma_n)\Gamma(\gamma_1+\ldots+\gamma_n)}{\Gamma(\beta-\alpha+\gamma_1+\ldots+\gamma_n)\Gamma(\beta+\gamma_1+\ldots+\gamma_n)\Gamma(\gamma_1)\ldots\Gamma(\gamma_n)}H^{0,\gamma+2:(\mu',\nu'+1);\ldots;\left(\mu^{(n)},\nu^{(n)}+1\right)}_{A+2,C+2:[B',D'];\ldots;\left[B^{(n)},D^{(n)}\right]}$$

$$\left(\begin{matrix}\left[(a):\theta',\ldots,\theta^{(n)}\right], & [1+\alpha-\beta-\gamma_1-\ldots-\gamma_n:\xi_1+\eta_1,\ldots,\xi_n+\eta_n], \\ \left[(c):\psi',\ldots,\psi^{(n)}\right], & [1+\alpha+m-\beta-\gamma_1-\ldots-\gamma_n:\xi_1+\eta_1,\ldots,\xi_n+\eta_n],\end{matrix}\right.$$

$$\begin{matrix}[1-\beta-\gamma_1-\ldots-\gamma_n:\xi_1+\eta_1,\ldots,\xi_n+\eta_n]:[(b'):\phi'],[1-\gamma_1:\xi_1];\ldots;\left[\left(b^{(n)}\right):\phi^{(n)}\right],[1-\gamma_n:\xi_n]; \\ [1-\gamma_1-\ldots-\gamma_n:\xi_1,\ldots,\xi_n]: \qquad [(d'):\delta'] \qquad ;\ldots; \qquad \left[\left(d^{(n)}\right):\delta^{(n)}\right];\end{matrix}$$

$$\left.u_1,\ldots,u_n\right),$$

provided that $Re(\beta-\alpha-m+\gamma_1+\ldots+\gamma_n)>0$, $Re(\gamma_i)>0$, ξ_i, η_i are any positive numbers, m is positive integer, and $|\arg u_i|<\frac{\pi}{2}\Delta_i$,

where

$$\Delta_i=-\sum_{j=y+1}^{A}\theta_j^{(i)}+\sum_{j=1}^{\nu^{(i)}}\phi_j^{(i)}-\sum_{j=\nu^{(i)}+1}^{B^{(i)}}\phi_j^{(i)}-\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{\mu^{(i)}}\delta_j^{(i)}-\sum_{j=\mu^{(i)}+1}^{D^{(i)}}\delta_j^{(i)}>0,\quad i=1,\ldots,n.$$

6. ANOTHER GENERALIZED MULTIDIMENSSIONAL LAGUERRE TRANSFORM.

In this section, we introduce the following new multidimenssional Laguerre Transform :

$$L^{(\beta_1,\ldots\beta_n;n_1,\ldots,n_n;K)}_{\gamma_1,\ldots,\gamma_n}\{\ \}=K\prod_{j=1}^{n}\frac{(-1)^{n_j}n_j!\Gamma(\gamma_j-\beta_j+n_j)}{\Gamma(\gamma_j-\beta_j)\Gamma(\gamma_j)} \tag{6.1}$$

$$\int_0^\infty\cdots\int_0^\infty e^{-\sum_{j=1}^{n}\left(\alpha_1^jx_1+\ldots+\alpha_n^jx_n\right)}\prod_{j=1}^{n}\left(\alpha_1^jx_1+\ldots+\alpha_n^jx_n\right)^{\gamma_j-1}L_{n_j}^{\beta_j}\left(\alpha_1^jx_1+\ldots+\alpha_n^jx_n\right)$$

$$\{\ \}dx_1\ldots dx_n,$$

where $Re(\gamma_j) > 0$, $Re(\gamma_j - \beta_j - n_j) > 0$, $n_j \in N$; $j = 1,\dots,n$

and $$K = \begin{vmatrix} \alpha_1^1 & \alpha_2^1 & \dots & \alpha_n^1 \\ \alpha_1^2 & \alpha_2^2 & \dots & \alpha_n^2 \\ \dots & \dots & \dots & \dots \\ \alpha_1^n & \alpha_2^n & \dots & \alpha_n^n \end{vmatrix} \neq 0$$

Here we derive

$$L_{\gamma_1,\dots,\gamma_n}^{(\beta_1,\dots,\beta_n,n_1,\dots,n_n;K)}\{1\} = 1, \tag{6.2}$$

$$L_{\gamma_1,\dots,\gamma_n}^{(\beta_1,\dots,\beta_n,n_1,\dots,n_n;K)}\left\{\left(\alpha_1^1 x_1+\dots+\alpha_n^1 x_n\right)^{m_1}\dots\left(\alpha_1^n x_1+\dots+\alpha_n^n x_n\right)^{m_n}\right\} \tag{6.3}$$

$$= \prod_{j=1}^{n}\frac{\left(\gamma_j-\beta_j\right)_{m_j}\left(\gamma_j\right)_{m_j}}{\left(\gamma_j-\beta_j-n_j\right)_{m_j}},$$

$$L_{\gamma_1,\dots,\gamma_n}^{(\beta_1,\dots,\beta_n;n_1,\dots n_n;K)}\left\{F_{C:D';\dots;D^{(n)}}^{A:B';\dots;B^{(n)}}\left(\begin{array}{l}\left[(a):\theta',\dots,\theta^{(n)}\right]:\left[(b'):\phi'\right];\dots;\left[\left(b^{(n)}\right):\phi^{(n)}\right];\\ \left[(c):\psi',\dots,\psi^{(n)}\right]:\left[(d'):\delta'\right];\dots;\left[\left(d^{(n)}\right):\delta^{(n)}\right];\end{array}\right.\right. \tag{6.4}$$

$$\left.\left. u_1\left(\alpha_1^1 x_1+\dots+\alpha_n^1 x_n\right)^{\xi_1},\dots,u_n\left(\alpha_1^n x_1+\dots+\alpha_n^n x_n\right)^{\xi_n}\right)\right\}$$

$$= F_{C:D'+1;\dots;D^{(n)}+1}^{A:B'+2;\dots;B^{(n)}+2}\left(\begin{array}{l}\left[(a):\theta',\dots,\theta^{(n)}\right]:\left[(b'):\phi'\right],\left[\gamma_1-\beta_1:\xi_1\right],\left[\gamma_1:\xi_1\right];\dots;\\ \left[(c):\psi',\dots,\psi^{(n)}\right] \quad : \left[(d'):\delta'\right],\left[\gamma_1-\beta_1-n_1:\xi_1\right];\dots;\end{array}\right.$$

$$\left.\begin{array}{l}\left[\left(b^{(n)}\right):\phi^{(n)}\right],\left[\gamma_n-\beta_n:\xi_n\right],\left[\gamma_n,\xi_n\right]\\ \left[\left(d^{(n)}\right):\delta^{(n)}\right], \quad \left[\gamma_n-\beta_n-n_n:\xi_n\right]\end{array} u_1,\dots,u_n\right)$$

provided that $Re(\gamma_i) > 0$, $Re(\gamma_i - \beta_i - n_i) > 0$, $n_i \in N$,

$$1-\xi_i + \sum_{j=1}^{C'} \psi_j^{(i)} + \sum_{j=1}^{D^{(i)}} \delta_j^{(i)} - \sum_{j=1}^{A} \theta_j^{(i)} - \sum_{j=1}^{B^{(i)}} \phi_j^{(i)} > 0;\ i = 1,\dots,n,$$

and $K = \begin{vmatrix} \alpha_1^1 & \alpha_2^1 & \dots & \alpha_n^1 \\ \alpha_1^2 & \alpha_2^2 & \dots & \alpha_n^2 \\ \dots & \dots & \dots & \dots \\ \alpha_1^n & \alpha_2^n & \dots & \alpha_n^n \end{vmatrix} \neq 0$

$$(6.5)\quad L_{\gamma_1,\dots,\gamma_n}^{(\beta_1,\dots,\beta_n;n_1,\dots,n_n;K)} \left\{ H_{A,C:[B',D'];\dots;[B^{(n)},D^{(n)}]}^{0,\lambda:(\mu',\nu');\dots;(\mu^{(n)},\nu^{(n)})} \left(\begin{matrix} \left[(a):\theta',\dots,\theta^{(n)}\right]:\left[(b'):\phi'\right];\dots;\left[\left(b^{(n)}\right):\phi^{(n)}\right]; \\ \left[(c):\psi',\dots,\psi^{(n)}\right]:\left[(d'):\delta'\right];\dots;\left[\left(d^{(n)}\right):\delta^{(n)}\right]; \end{matrix} \right. \right.$$

$$\left. \left. u_1\left(\alpha_1^1 x_1+\dots+\alpha_n^1 x_n\right)^{\xi_1},\dots u_n\left(\alpha_1^n x_1+\dots+\alpha_n^n x_n\right)^{\xi_n} \right) \right\}$$

$$= H_{A,C:\left[B^1+2,D^1+1\right];\dots;\left[B^{(n)}+2,D^{(n)}+1\right]}^{0,\lambda:(\mu^1,\nu^1+2);\dots;(\mu^{(n)},\nu^{(n)}+2)} \left(\begin{matrix} \left[(a):\theta',\dots,\theta^{(n)}\right]:\left[(b'):\phi'\right], \\ \left[(c):\psi',\dots\psi^{(n)}\right]:\left[(d'):\delta'\right], \end{matrix} \right.$$

$$\left. \begin{matrix} [1-\gamma_1+\beta_1:\xi_1],[1-\gamma_1:\xi_1];\dots;\left[\left(b^{(n)}\right):\phi^{(n)}\right],[1-\gamma_n+\beta_n:\xi_n],[1-\gamma_n:\xi_n]; \\ [1-\gamma_1+\beta_1+n_1:\xi_1];\dots;\left[\left(d^{(n)}\right):\delta^{(n)}\right],[1-\gamma_n+\beta_n+n_n:\xi_n]; \end{matrix} \; u_1,\dots,u_n \right)$$

provided that $Re(\gamma_i) > 0$, $Re(\gamma_i - \beta_i - n_i) > 0$, $n_i \in N$, $|\arg u_i| < \frac{\pi}{2}\Delta_i$,

where

$$\Delta_i = -\sum_{j=\gamma+1}^{A} \theta_j^{(i)} + \sum_{j=1}^{\nu^{(i)}} \phi_j^{(i)} - \sum_{j=\nu^{(i)}+1}^{B^{(i)}} \phi_j^{(i)} - \sum_{j=1}^{C} \psi_j^{(i)} + \sum_{j=1}^{\mu^{(i)}} \delta_j^{(i)} - \sum_{j=\mu^{(i)}+1}^{D^{(i)}} \delta_j^{(i)} > 0,\ i = 1,\dots,n$$

and $K = \begin{vmatrix} \alpha_1^1 & \alpha_2^1 & \dots & \alpha_n^1 \\ \alpha_1^2 & \alpha_2^2 & \dots & \alpha_n^2 \\ \dots & \dots & \dots & \dots \\ \alpha_1^n & \alpha_2^n & \dots & \alpha_n^n \end{vmatrix} \neq 0$

Special Cases

(6.6) $$L_{\gamma_1,\dots,\gamma_n}^{(\beta_1,\dots,\beta_n,n_1,\dots,n_n;K)}$$

$$\left\{\psi_2^{(n)}\left(a;\gamma_1-\beta_1,\dots,\gamma_n-\beta_n;u_1\left(\alpha_1^1x_1+\dots+\alpha_n^1x_n\right),\dots,u_n\left(\alpha_1^nx_1+\dots+\alpha_n^nx_n\right)\right)\right\}$$

$$= F_A^{(n)}(a,\ \gamma_1,\dots,\gamma_n;\ \gamma_1-\beta_1-\mathrm{n}_1,\dots,\gamma_n-\beta_n-n_n;\ u_1,\dots,u_n)$$

provided that $|u_1| + \dots + |u_n| < 1$ and all conditions of (6.1) are satisfied.

(6.7) $$L_{\gamma_1,\dots,\gamma_n}^{(\beta_1,\dots,\beta_n,n_1,\dots,n_n;K)}\left\{\psi_2^{(n)}\left(a,\gamma_1,\dots,\gamma_n;u_1\left(\alpha_1^1x_1+\dots+\alpha_n^1x_n\right),\dots,u_n\left(\alpha_1^nx_1+\dots+\alpha_n^nx_n\right)\right)\right\}$$

$$= F_A^{(n)}(a,\ \gamma_1-\beta_1,\dots,\gamma_n-\beta_n;\ \gamma_1-\beta_1-n_1,\dots,\gamma_n-\beta_n-n_n;\ u_1,\dots,u_n),$$

valid if all conditions of (6.6) are satisfied.

(6.8) $$L_{\gamma_1,\dots,\gamma_n}^{(\beta_1,\dots,\beta_n,n_1,\dots,n_n;K)}$$

$$\left\{\phi_2^{(n)}\left(\gamma_1-\beta_1-n_1,\dots,\gamma_n-\beta_n-n_n;a;u_1\left(\alpha_1^1x_1+\dots+\alpha_n^1x_n\right),\dots,u_n\left(\alpha_1^nx_1+\dots+\alpha_n^nx_n\right)\right)\right\}$$

$$= F_B^{(n)}(\gamma_1,\dots,\gamma_n,\ \gamma_1-\beta_1,\dots,\gamma_n-\beta_n;\ a;\ u_1,\dots,u_n).$$

Provided that $|u_i| < 1$, $i = 1,\dots,n$ and all conditions of (6.1) are satisfied.

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BILINEAR MAPS ON FUZZY NORMED LINEAR SPACES

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ABSTRACT : In this paper we have introduced several types of continuity of bilinear maps on a fuzzy normed linear space. Furthermore, Relations between pointwise continuity and separate continuity and between hypocontinuity and boundedness are also established. The results obtained serve to demonstrate that it is possible to establish a theory of these objects along the line pursue here with considerable hope for success.

Key words : Fuzzy normed linear space, bilinear map, Fuzzy bounded set, Fuzzy hypocontinuous mapping.

1. PRELIMINARIES

In this section we recall some basic definitions and results. For details, we refer to [1], [2].

Definition 1.1. A fuzzy real number η is a Fuzzy set on $\mathbb{R}$. That is, $\eta : \mathbb{R}$ to $I = [0, 1]$.

Since each $r \in \mathbb{R}$ can be considered a fuzzy real number $\bar{r}$ defined by

$$\bar{r}(t) = \begin{cases} 1 \text{ if } t = r \text{ and} \\ 0 \text{ if } t \neq r. \end{cases}$$

$\mathbb{R}$ can be embedded in $\mathbb{R}(I)$.

Definition 1.2. A fuzzy real number η is called nonnegative if $\eta(t) = 0$ for all $t < 0$. The set of all nonnegative fuzzy real numbers is denoted by $\mathbb{R}^*(I)$.

Proposition 1.3. The arithmetic operations +, – and/on $\mathbb{R}(I) \times \mathbb{R}(I)$ are defined by

(i) $(\eta_1 \oplus \eta_n)(t) = \sup\limits_{s \in \mathbb{R}} \min (\eta_1(s), \eta_2(t - s)), t \in \mathbb{R}$

(ii) $(\eta_1 \ominus \eta_2)(t) = \sup\limits_{s \in \mathbb{R}} \min (\eta_1(s), \eta_2(t - s)), t \in \mathbb{R}$

(iii) $(\eta_1 \odot \eta_2)(t) = \sup\limits_{\substack{s \in \mathbb{R} \\ s \neq 0}} \min (\eta_1(s), \eta_2(t/s)), t \in \mathbb{R}$

(iv) $(\eta_1/\eta_2)(t) = \sup\limits_{s\in\mathbb{R}} \min\,(\eta_1(ts),\ \eta_2(s)),\ t\in\mathbb{R}$

The additive and multiplicative identities in $\mathbb{R}(I)$ are $\bar{0}$ and $\bar{1}$ respectively.

Where, $\bar{0}(t) = \begin{cases} 1 \text{ if } t = 0 \\ 0 \text{ if } t \neq 0 \end{cases}$ and $\bar{1}(t) = \begin{cases} 1 \text{ if } t = 1 \\ 0 \text{ if } t \neq 1. \end{cases}$

Let $\Theta\eta$ be defined as $\bar{0}\,\Theta\eta$. From (i) & (ii) it follows that $(-\eta)(t) = \eta(-t)$ for all $t\in\mathbb{R}$ and $\eta_1\Theta\eta_2 = \eta_1 \oplus (\Theta\eta_2)$.

Definition 1.4. The α - level set of a fuzzy real number η denoted by $[\eta]_\alpha$, $0 < \alpha \leq 1$ is defined as $|\eta|_\alpha = \{t|\eta(t) \geq \alpha\}$.

Definition 1.5. Let X be a linear space over the field $\mathbb{R}$. Let II.II: $\mathbb{R}$ to $\mathbb{R}^*$ (I) and let the mappings L, R: [0, 1] × [0, 1] to [0, 1] be symmetric, nondecreasing in both arguments and satisfy $L(0, 0) = 0$ and $R(1, 1) = 1$. Write $[\mathrm{II}\,x\,\mathrm{II}]_\alpha = [\mathrm{III}\,x\,\mathrm{III}_1^\alpha, \mathrm{III}\,x\,\mathrm{III}_2^\alpha]$, for $x\in X$, $0 < \alpha \leq 1$ and suppose for all $x\in X$, $x \neq 0$, there exists $\alpha_0 \in [0, 1]$ (independent of x) such that for all $\alpha \leq \alpha_0$.

(A) $\mathrm{III}\,x\,\mathrm{III}_2^\alpha < \infty$

(B) Inf $\mathrm{III} \times \mathrm{III}_1^\alpha > 0$

α

The quadruple (X, II.II, L, R) is called a fuzzy normed linear space and II.II a fuzzy norm, if

(i) II x II = $\bar{0}$ iff $x = 0$

(ii) II r x II = I r I II x II, $x\in X$, $r\in\mathbb{R}$

(iii) for all x, $y \in X$

(a) Wherever,

s ≤ III X III_1^1, t ≤ III yIII_1^1 and s + t ≤ IIIx+y III_1^1, IIx + yII (s + t) ≥ L (IIxII(s), IIyII(t))

(b) Whenever,

s ≥ III X III_1^1, t ≥ III yIII_1^1 and s + t ≥ IIIx+y III_1^1, IIx + yII (s + t) ≤ R (IIxII(s), IIyII(t))

Remarks 1.6. In the sequal, we take $L(x, y) = \min(x, y)$ and $R(x, y) = \max(x, y)$ for $x, y \in [0, 1]$ and in this case we write $(X, \|.\|)$ or simply X.

Proposition 1.7. In a fuzzy normed linear space $(X, \|.\|, \min, \max)$ the triangular inequality (iii) of definition 1.5 is equivalent to $\|x + y\| \leq \|x\| + \|y\|$.

Definition 1.8. Let $(X, \|.\|)$ be a fuzzy normed linear space. For fixed $\alpha \in (0, 1]$, $\in > 0$, the fuzzy set $S_\alpha(x, \in)$ defined by $S_\alpha(x, \in)(y) = \begin{cases} \alpha \text{ iff } \|y - x\|_2^\alpha < \in \\ 0, \text{ otherwise} \end{cases}$

is said to be an α - open sphere in $(X, \| \|)$

Definition 1.9. Let $(X, \| \|_1)$ and $(Y, \| \|_2)$ be two fuzzy normed linear spaces and F : X to Y be linear mapping from X into Y. We take $L(x, y) = \min(x, y)$ and $R(x, y) = \max(x, y)$.

F is defined to be fuzzy continuous at $x_0 \in X$ if for $\in > 0$ there exists $\delta > 0$ such that $\| F(x) - F(x_0) \|_2 < \overline{\in}$ whenever $\|x - x_0\|_1 < \overline{\delta}$.

Definition 1.10. If A_1 and A_2 are two fuzzy subsets of X and Y respectively then the Cartesian product $A_1 \times A_2$ of fuzzy subsets A_1 and A_2 is a fuzzy subset of $X \times Y$ defined by $(A_1 \times A_2)(x, y) = \min\{A_1(x), A_2(y)\}$ for each pair $(x, y) \in X \times Y$.

Proposition 1.11. If $(X, \|.\|_X)$ and $(Y.\|.\|_Y)$ are two fuzzy normed linear spaces, then $X \times Y$ is also a fuzzy normed linear space with the norm defined on $X \times Y$ by

$$\| (x, y) \|_{X \times Y} = \| X \|_X \oplus \|y\|_Y, \; x \in X, y \in Y.$$

2. MAIN RESULTS

Proposition 2.1. Let $(X, \|.\|_x)$, $(Y, \|.\|_Y)$, $(Z, \|.\|z)$ be three fuzzy normed linear spaces. Then the bilinear map $F : X \times Y$ to Z is continuous on the product space $X \times Y$ if it is continuous at the point (0, 0).

Proof. Let F be continuous at (0, 0). Then by the definition of continuity for $\in > 0$ there exists $\delta > 0$ such that $\|F(x, y) - F(0, 0)\|_Z < \overline{\in}$ whenever $\| (x, y) - (0, 0) \|_{X \times Y} < \overline{\delta}$.

That is $\| F(x, y) \|z < \overline{\in}$ whenever $\|(x, y)\|_{X \times Y} < \overline{\delta}$,

So that, $\| F(x, y) \|z < \overline{\in}$ whenever $\| x \|_X \oplus \| y \|_Y < \overline{\delta}$,

Next, we show that F is continuous on $X \times Y$.

Let $(x_0, y_0) \in X \times Y$. We have to show that $\| F(x_0 + h, y_0 + k) - F(x_0, y_0)\|_Z < \overline{\in}$.

Whenever $\| (x_0 + h, y_0 + k) - (x_0, y_0) \|_{X \times Y} < \bar{\delta}$.

That is, $\|(h, k)\|_{X \times Y} < \bar{\delta}$, or equivalently,

$\| h\|_X \oplus \| k \|_y < \bar{\delta}$ ....(A)

Now we have,

$F(x_0 + h, y_0 + k) = F(x_0 y_0) + F(x_0, k) + F(h, y_0) + F(h, k)$,

So, $\| F(x_0 + h, y_0 + k) - F(x_0 y_0) \|_z$

$= \| F (x_0, k) + F(h, y_0) + F(h, k) \|_z$

$\leq \| F(x_0, k) \|_z + \| F(h, y_0) \|_z + \| F(h, k) \|_z$

$< \bar{\in} / 3 + \bar{\in} / 3 + \bar{\in} / 3$

Whenever $\|x_0 \|_x \oplus \| k \|_Y < \bar{\delta}_1$, $\| h \|_X \oplus \|y_0\|_Y < \bar{\delta}_2$ and $\| h \|_X \oplus \| k \|_Y < \bar{\delta}_3$. $(\delta_1 > 0,\ \delta_2 > 0,\ \delta_3 > 3)$ and

if $\delta = \text{Min}\ \{\delta_1, \delta_2, \delta_3\}$

Then from (A), if follows that $\|F (x_0 + h, y_0 + k) - F(x_0, y_0)\|_Z < \bar{\in}$ whenever $\| h\|_X \oplus \| k\|_Y < \bar{\delta}$. Consequently $F : X \times Y$ to Z is continuous everywhere.

Proposition 2.2. Every continuous bilinear map is separately continuous.

Proof. Let $F : X \times Y$ to Z be a continuous bilinear map where $(X \|.\|_X)$, $(Y, \|.\|_Y)$ and $(Z, \|.\|_Z)$ are fuzzy normed linear spaces. Therefore F is continuous at every point of $X \times Y$ and, in particular, it is continuous at (0, 0). So, for $\in > 0$, there exists $\delta > 0$ such that $\|F(x, y) \|_Z < \bar{\in}$.

Whenever $\|(x, y) \|_{X \times Y} < \bar{\delta}$

That is, $\|x\|_X \oplus \|y\|_Y < \bar{\delta}$ .....(*)

We need to prove that F is separately continuous. That is, for every $y \in Y$ the linear map $F(\cdot\,, y)$: x to $F(x, y)$ from X into Z is continuous and for every $x \in X$ the linear map

$F(x, \cdot)$: y to $F(x, y)$ from Y into Z is continuous

Now, $F (x_0 + h, y) = F(x_0, y) + F(h, y)$

So, $\|F(x_0 + h, y) - F(x_0, y)\|_z = \|F(h, y)\|_z < \bar{\in}$ whenever $\|h\|_X < \bar{\delta}$ $(\delta > 0)$

Thus, for $\in > 0$, there exists $\delta > 0$ such that $\| F(x_0 + h, y) - F(x_0, y)\|_Z < \bar{\in}$ whenever $\| h \|_x < \bar{\delta}$ and so the linear map from X to Z is continuous.

Again we have

$F(x, y_0 + k) = F(x, y_0) + F(x, k)$

So, $\|F(x, y_0 + k) - F(x, y_0)\|_Z = \|F(x, k)\|_Z < \bar{\epsilon}$ whenever $\| k\|_Y < \bar{\delta}$.

Thus, $\|F(x, y_0 + k) - F(x, y_0)\|_z < \bar{\epsilon}$ whenever $\|k\|_Y < \bar{\delta}$ and consequently the linear map from Y to Z is continuous. Hence the result follows.

Proposition 2.3. Let $(X, \|.\|_x)$, $(Y, \|.\|_Y)$ and $(Z, \|.\|_Z)$ be three fuzzy normed linear spaces. Thus every separately continuous bilinear map from X × Y to Z is continuous.

Proof. Let F: X × Y to Z be a bilinear map which is separately continuous.

Thus for $\epsilon_1 > 0$, $\epsilon_2 > 0$ there exists $\delta_1 > 0$, $\delta_2 > 0$ such that

$\|F(x_0 + h, y) - F(x_0, y)\|_Z < \bar{\epsilon}_1$ whenever $\|(x_0 + h, y) - (x_0, y)\|_{X \times Y} < \bar{\delta}_1$,

That is, $\| h \|_x < \bar{\delta}_1$ ..... (1)

and $\|F(x, y_0 + k) - F(x, y_0)\|_z < \bar{\epsilon}_2$ whenever $\|(x, y_0 + k) - (x, y_0)\|_{X \times Y} < \bar{\delta}_2$,

That is $\| k \|_Y < \bar{\delta}_2$ ..... (2)

Now, we need to prove that F: X × Y to Z is continuous. To prove this it is sufficient to prove that F: X × Y to Z is continuous at (0, 0).

That is, we must show that for $\epsilon > 0$ there exists $\delta > 0$ such that $\| F(x, y) \|_Z < \bar{\epsilon}$ whenever $\|(x, y)\|_{X \times Y} < \bar{\delta}$, or equivalently $\| x \|_X \oplus \|y\|_Y < \bar{\delta}$,

Now, we need to prove that F: X × Y to Z is continuous. To prove this it is sufficient to prove that F: X × Y to Z is continuous at (0, 0).

That is, we must show that for $\epsilon > 0$ there exists $\delta > 0$ such that $\| F (x, y) \|_Z < \bar{\epsilon}$ whenever $\|(x, y)\|_{X \times Y} < \bar{\delta}$, or equivalently $\| x \|_X \oplus \|y\|_Y < \bar{\delta}$,

Now, we have $F(x, y) = F(x + h, y) - F(h, y)$

Therefore, $\|F(x, y)\|_Z = \|F(x + h, y) - F(h, y)\|_Z < \bar{\epsilon}_1$ whenever $\| x \|_X < \bar{\delta}_1$

Again, we have $F(x, y) = F(x, y + k) - F(x, k)$

Therefore $\|F(x, y)\|_Z = \|F(x, y + k) - F(x, k)\|_z < \bar{\epsilon}_2$ whenever $\|y\|_Y < \bar{\delta}_2$

If $\epsilon = \max \{\epsilon_1, \epsilon_2\}$ and $\delta = \min\{\delta_1 \delta_2\}$

then it follows that

$\|F(x, y)\|_Z < \bar{\epsilon}$ whenever $\| x \|_X < \bar{\delta}$ and $\| y \|_Y < \bar{\delta}$, that is, $\| x \|_X \oplus \|y\|_Y < \bar{\delta}$ Thus, since F is continuous at (0, 0), F is a continuous bilinear map from X × Y to Z.

Definition: 2.4. Let $(X, \|\;\|_X)$ be a fuzzy normed linear space. A fuzzy subset μ of X is called fuzzy bounded if $\mu(x) \leq k\ S_\alpha(0, 1)(x)$ for all x, $k > 0$.

Definition: 2.5. Let $(X, \|\;\|_X)$, $(Y, \|\;\|_Y)$ and $(Z, \|\;\|_Z)$ be three fuzzy normed linear spaces. Let G be a collection of fuzzy bounded subsets of X. Now a bilinear map b: X × Y to Z is called fuzzy hypocontinuous with respect to G (or G –f-hypocontinuous) if it is separately continuous and satisfies the condition that for every neigbourhood W of 0 in Z and $A(x) < G(x)$ for all x, there exists a neighbourhood V of 0 in Y such that $A(x) > 0$, $V(y) > 0$ imply $W(b(x, y)) > 0$.

Proposition 2.6. Let $(X, \|\;\|_X)$, $(Y, \|\;\|_Y)$ and $(Z, \|\;\|_Z)$ be three fuzzy normed linear spaces. G a collection of fuzzy bounded subsets of X and b: X × Y to Z a fuzzy hypocontinuous bilinear map with respect to G. If $A(x) < G(x)$ for all x such that $X(x) > 0$ and B a fuzzy bounded subset of Y, then

(i) The image of A × B under the map b is fuzzy bounded in Z.

(ii) The map b is continuous on A × Y

Proof. (i) Given that A is a fuzzy bounded subset of X and B a fuzzy bounded subset of Y, we need to prove that $b(A \times B)$ is fuzzy bounded in Z.

Now, A is fuzzy bounded in X so for $\alpha \in (0, 1]$ and for all x such that $X(x) > 0$, $k > 0$

$$A(x) < k\ S_\alpha(0, 1)(x) = S_\alpha(0, k)(x) \quad \text{..... (1)}$$

Also B is fuzzy bounded in Y, so for $\beta \in (0, 1]$ and for all y such that $Y(y) > 0$, $k_1>0$,

$$B(y) < k_1 S_\beta(0, 1)(y) = S_\beta(0, k_1)(y) \quad \text{..... (2)}$$

We need to show that for, $\delta \in (0, 1]$ and for all z such that $Z(z) > 0$, $k_2 > 0$

$$b(A \times B)(z) < k_2\ S_\delta(0, 1)(z) = S_\delta(0, k_2)(z) \quad \text{..... (3)}$$

Now, since b is G-f-hypocontinuous, then by definition $A(x) > 0$, $S_\beta(0, k_1)(y) > 0$ imply

$$S_\delta(0, k_2)(b(x, y)) > 0 \quad \text{..... (4)}$$

Since $B(y) < S_\beta(0, k)(y)$, for all y such that $Y(y) > 0$.

Thus (4) can be stated as : $A(x) > 0$ and $B(y) > 0$ imply $S_\delta(0, k_2)\ (b(x, y)) > 0$,

Since $b(A \times B) = \{b(x, y) : A(x) > 0, B(y) > 0\}$,

Therefore, $b(A \times B)(z) < S_\delta(0, k_2)(z)$ for all z such that $Z(z) > 0$ and consequently $b(A \times B)$ is fuzzy bounded.

(ii) Let $(A \times Y)(x_0, y_0) > 0$. We need to show that given $\in > 0$, there exists $\delta > 0$ such that $\|b(x_0 + h, y_0 + k) - b(x_0, y_0)\|_Z < \bar{\in}$ whenever $\|(x_0 + h, y_0 + k) - (x_0, y_0)\|_{X\times Y} < \bar{\delta}$

that is, $\| h\|_X \oplus \| k \|_Y < \bar{\delta}$.

Now,

$b(x_0 + h, y_0 + k) = b(x_0, y_0) + b(x_0, k) + b(h, y_0) + b(h, k)$

Therefore,

$\| b(x_0 + h, y_0 + k) - b(x_0, y_0)\|_Z = \|b(x_0, k) + b(h, y_0) + b(h, k)\|_Z$

$\leq \|b(x_0, k)\|_Z + \|b(h, y_0)\|_Z + \|b(h, k)\|_Z$ .... (1)

Since b is separately continuous and thus continuous on $X \times Y$ it is continuous at (0, 0) also.

Therefore,

$\|b(x_0, k)\|_Z < \bar{\in}/3$ whenever $\|x_0\|_X \oplus \| k \|_Y < \bar{\delta}_1$.

$\|b(h, y_0)\|_Z < \bar{\in}/3$ whenever $\| h \| x \oplus \| y_0 \|_Y < \bar{\delta}_2$, and

$\|b(h, k)\|_Z < \bar{\in}/3$ whenever $\| h \|_X \oplus \| k \|_Y < \bar{\delta}_3$.

Thus from (1) we have,

$\|b(x_0 + h, y_0 + k) - b(x_0, y_0)\|_Z < \bar{\in}/3 + \bar{\in}/3 + \bar{\in}/3 = \bar{\in}$,

Whenever

$\|x_0\|_x \oplus \| k \|_Y < \bar{\delta}_1$, $\| h \|_X \oplus \| y_0 \|_Y < \bar{\delta}_2$, $\| h \| x \oplus \| k \|_Y < \bar{\delta}_3$.

If $\delta = \min\{\delta_1, \delta_2, \delta_3\}$

Then it follows that

$\| b(x_0 + h, y_0 + k) - b(x_0, y_0)\|_z < \bar{\in}$ whenever $\| h \|_X \oplus \| k \|_Y < \bar{\delta}$.

b is continuous on $A \times Y$ and so the proof is complete.

Conclusion. We have shown that it is possible to define bilinear maps on fuzzy normed linear spaces in such a way as to obtain several of the standard basic but nevertheless important results needed to establish a mature theory. Although we have not yet produced such a mature theory we have shown that with the structure developed sofar it should prove possible to do so in the future.

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OPTIMIZATION UNDER CONGESTION AND GEAR SATURATION

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ABSTRACT : The paper considers the exploitation of two ecologically independent species given by Logistic growth models, where the total harvest satisfies modified Cobb-Douglass relation. It is verified that the interior equilibrium of the exploited model is stable, and that there is no limit cycle. Next the concept of bionomic equilibrum is applied to calculate the threshold values of cost per unit effort and the maximum value of the effort. With the knowledge of the control set, thus obtained from the above value of the maximum effort, the total revenue is optimized under two types of depreciations of which one is the standard one. Further, all the results are verified by suitable choice of the parameters. Lastly, a few observations are made on some of the results obtained in this paper.

Key Words : Cobb-Douglass relation, Modified Cobb-Douglass relation, Bionomic equilibrnum, Threshold values of cost per unit effort.

1. INTRODUCTION

Congestion and gear saturation are two common phenomena in fishery problems. Whenever there is a huge concentration of biomasses at certain region of the sea, there occurs naturally, a competition amongst different fisheries to exploit in that region. This leads to the congestion of the vessels used, and ultimately to a huge reduction of biomasses. In order to avoid this situation, some restrictions are made on the effort of fishing E. Again, it is a very common experience, that there is always an upper bound to the capacity of the net (called gear saturation), which depends on the mesh size parameter of the net. Hence some restrictions are to be imposed on the biomass x of the total harvest h. The usual catch per unit effort hypothesis h = qEx (q being the catchability coefficient) does not work in the above context. This necessitates the adoption of Cobb-Douglass relation h = $qE^{\alpha}x^{\beta}$ $(0 < \alpha < 1, 0 < \beta < 1)$. As a result, the production function h(x, E) shows decreasing marginal returns in both the input variables x and E. As α is arbitrary, so under suitable choice of α, the total harvest h(x, E) remains bounded with respect to E and x, even if $\beta = 2$. We call h = $qE^{\alpha}x^{2}$ $(0 < \alpha < 1)$, the modified Cobb-Douglass relation.

Although it is more realistic to consider h = $qE^{\alpha}x^{\beta}$ in place of h= qEx, still in the literature, no attempt is made to consider such general expression of h in calculating the optimal value of the revenue. This is due to the fact, that the calculations become too difficult in this case. However, we show, in the present paper, that the whole analysis may be carried out smoothly with h = $qE^{\alpha}x^{2}$, $0 < \alpha < 1$.

Bhattacharya and Begum [2] considered h = qEx and denoted by L and $\pi = 0$, the locus of dynamic equilibrium and the zero profit line respectively. They defined a fasible bionomic equilibrium (x_{∞}, y_{∞}) as the point of intersection of L = 0 and $\pi = 0$, provided the point occured in the positive quadrant of the XY-plane, and the effort remained positive at that point. They also defined two partially feasible bionomic equilibria (x_{∞}, 0) and (0, y_{∞}) as the point of intersections of L with positive x-axis and positive y-axis respectively, provided the effort E and the revenue π remained positive at those points. They showed, for the first time, that the concept of bionomic equilirbium could be used to determine the thershold values of the cost per unit effort, and the maximum value of the effort. They also pointed out that, unless cost per unit effort c was kept under proper threshold values, feasible or even partially feasible bionomic equilibrium could not occur. Therefore, it would lead to non-determination of E_{max} or sup (E_{max}), as the case might be. As a result, optimization of revenue would lose its meaning.

Normally L = 0 and $\pi = 0$ are both straight lines. But if none of them is a straight line, then consideration of bionomic equilibrium deserves special attention. Our present model is an example on this point. First of all, we obtain, as in [1], threshold values of cost per unit effort, and the maximum value of the effort. Next, using the standard depreciation $e^{-\delta t}$, we obtain the optimal path, the optimal biomasses and the optimal effort. Lastly, similar results are obtained using a depreciation which is different from the standard one.

2. PROBLEM FORMULATION

Let the exploited model be taken as

$$\frac{dx}{dt} = rx\left(1-\frac{x}{K}\right) - q_1E'x^2$$

$$\frac{dy}{dt} = sy\left(1-\frac{y}{L}\right) - q_2E'y^2 \qquad \text{...(I)}$$

with restrictions (i) a > b, or (ii) a < b where a = rLq_2, b = sKq_1, $E' = E^{\alpha}$ ($0 < \alpha < 1$) (E being the total effort exerted); q_1, q_2 are the catchability coefficients. If p_1, p_2 are the selling prices of unit biomasses of x and y respectively; then the following problems are considerd :

(i) To find the maximum threshold value of the cost per unit effort c and the maximum value $E_{max}/(\text{Sup}(E_{max})$ of the effort E.

(ii) To find the optimal values of the revenues given by

$$J = \int_0^\infty e^{-\delta t}\left(p_1q_1x^2 + p_2q_2y^2 - c'\right)E' \text{ dt., } \delta > 0,\ E'(0) = E_0 \qquad \text{.....II(A)}$$

$$\text{and } J = \int_{t_0}^\infty \frac{1}{t^{\delta+2}}\left(p_1q_1x^2 + p_2q_2y^2 - c'\right)E' \text{ dt, } \delta > 0,\ E'(t_0) = E_{t_0} \qquad \text{.....II(B)}$$

separately, where c′ is the cost for which c′E′ = cE, and further in case II(B), $(p_1q_1x^2 + p_2q_2y^2 - c')\,E'(t) = 0(t^\delta)$.

3. STABILITY OF THE SYSTEM

The only non-trivial equilibrium $((\bar{x})e, (\bar{y})e)$ of the system (I) is found to be $(\bar{x})_e = \frac{rK}{r + q_1E'L}$, $(\bar{y})_e = \frac{sL}{s + q_2E'L}$.

It is easily shown that $((\bar{x})_e, (\bar{y})_e)$ is an asymptotically stable node. Further, the non-linear system (I) does not possess any limit cycle. In fact, by applying Bendixon-De Luc criteria with $B(x, y) = \frac{1}{x^2y^2}$, it is proven that $\frac{\partial}{\partial x}(BF) + \frac{\partial}{\partial y}(BG)$ does not change sign in the first quadrant of the phase plane.

4. MAIN THEOREMS

Theorem 4.1. Let the exploited two species system be given by (I). Then for this system no partially feasible bionomic equilibrium exists. But feasible bionomic equilibrium (x_∞, y_∞) always occurs under both the cases a < b and a > b, if the upper thershold value c'_{max} of c′ is equal to

$$p_1q_1k^2 + p_2q_2L^2 \qquad \text{..... (4.1)}$$

Further, for each such threshold value of c′, $0 < c' < c'_{max}$, the maximum value E'_{max} cf E′ is determined by the unique positive root of the equation,

$$A_1(E')^4 + A_2(E')^3 - A_3(E')^2 - A_4(E') - A_5 = 0 \qquad \text{..... (4.2)}$$

where

$$A_1 = c'q_1^2q_2^2K^2L^2,\ A_2 = 2c'q_1q_2KL\ (sKq_1 + rLq_2)$$

$A_3 = p_1 q_1 q_2^2 r^2 K^2 L^2 + p_2 q_1^2 q_2 s^2 K^2 L^2 - c'(sKq_1 + rLq_2)^2 - 2rsq_1q_2KL$

$A_4 = 2[p_1q_1q_2r^2sK^2L + p_2q_1q_2rs^2KL^2 - rs(sKq_1 + rLq_2)]$

$A_5 = p_1q_1k^2 + p_2q_2L^2 - c'$

Sup (E'_{max}) is calculated by considering $c' < c'_{max}$.

Lastly, E_{max} is determined (in principal value) from E'_{max} and the upper threshold value c_{max} of c is determined from the relation $cE = c'E'$.

Proof. Dynamic equilibrium of the exploited system (I) is given by, $((\bar{x})_e, (\bar{y})_e) = \left(\frac{rK}{r + Kq_1E'}, \frac{sL}{s + Lq_2E'}\right)$. Naturally, that of the corresponding unexploited system is given by $(\bar{x}, \bar{y}) = (K, L)$. Now for all points $(x, y) \in L' = 0$, for which $E' > 0$, it is seen that $x \neq 0$, $y \neq 0$. Hence, partially feasible bionomic equlibrium can never occur.

Next it is seen (figure 1, 2) that the locus of $((\bar{x})_e, (\bar{y})_e)$ is a hyperbola passing through the origin and its equation is given by

$$L' : (a - b)xy + bLx - aKy = 0 \qquad \text{..... (4.3)}$$

Also it is found that the asymptotes of this hyperbola are $x = \alpha$, $y = \beta$, where $\alpha = \left(\frac{aK}{a-b}\right)$ and $\beta = -\left(\frac{bL}{a-b}\right)$.

Again, (figure 1, 2) the zero profit curve $\pi' = 0$ is given by the ellipse

$$\frac{x^2}{(c'/p_1q_1)} + \frac{y^2}{(c'/p_2q_2)} = 1, \qquad \text{..... (4.4)}$$

Further, the intersection of L' and $\pi' = 0$ is exactly one point in the positive quadrant in both the cases $a > b$ and $a < b$ (fig. 1 and Fig. 2).

Again comparing $(\bar{x}, \bar{y})$ and $((\bar{x})_e, (\bar{y})_e)$, it is found that any point $(x, y) \in L'$, satisfies $x < K$ and $y < L$, $\forall\ E' > 0$. Obviously (K, L) must be a point outside the ellipse, because π' is negative for all points inside the ellipse given by (4.4). For the same reason, E'_{max} is equal to that value of E, which corresponds to the point of intersection of L' and $\pi' = 0$. Thus E'_{max} is finally determined as stated in the theorem. Again, for feasible bionomic equilibrium, $\pi' > 0$, $E' > 0$, So, $c' < p_1q_1x^2 + p_2q_2y^2 < p_1q_1K^2 + p_2q_2L^2$. This determines the upper threshold value of c'. Moreover, for $0 < c' < c'_{max}$, feasible bionomic equilibrium (x_∞, y_∞) occurs at A and B in the cases $a > b$ and $a < b$ respectively.

Theorem 4.2. Let the exploited system be given by (I). Let the total discounted revenue be given by II(A). Then the optimal path is given by

$$p_1q_1x^2 + p_2q_2y^2 - c' - 2\left\{\frac{p_1q_1r(K-x)x^2}{K(\delta+r)} + \frac{p_2q_2s(L-y)y^2}{L(\delta+s)}\right\} = 0 \qquad \text{..... (4.5)}$$

Further, the optimal biomass y_δ is determined by the unique positive root of the equation,

$$\begin{aligned}
&2p_2q_2s(\delta + r)(a - b)^3y^6 + p_2q_2L(\delta + r)(a - b)^2(a\delta - as - b\delta + 7bs)y^5\\
&\quad + 3p_2q_2L^2b(a - b)(\delta + r)(a\delta - b\delta - as + 3bs)y^4\\
&\quad + L[p_2q_2L^2b^2(\delta + r)(3a\delta - 7b\delta - 3as + 5bs)\\
&\quad + p_1q_1K^2a^2(\delta + s)(a\delta + ar - b\delta + br)\\
&\quad - c'(\delta + r)(\delta + s)(a - b)^3]y^3 + L^2b[p_2q_2L^2b^2(\delta + r)(\delta - s)\\
&\quad + p_1q_1K^2a^2(\delta + s)(\delta - r) - 3c'(\delta + r)(\delta + s)(a - b)^2]y^2\\
&\quad - 3c'L^3b^2(a - b)(\delta + r)(\delta + s)y - c'L^4(\delta + r)(\delta + s)b^3 = 0 \qquad \text{.....(4.6)}
\end{aligned}$$

and the optimal biomass x_δ is given by the positive root of the equation $(a - b)xy_\delta + bLx - aKy_\delta = 0$.

Lastly, optimal effort $(E')^*$ is given by $y_\delta = \left(\frac{sL}{s+Lq_2(E')^*}\right)$ and the optimal effort E^* is determined (in principal value) from $(E')^*$.

Proof. For convenience, we write $F(x) = rx\left(1-\frac{x}{K}\right)$, $G(y) = sy\left(1-\frac{y}{L}\right)$. Then the Hamiltonian corresponding to (I) and II(A) is given by,

$$\begin{aligned}
H &= e^{-\delta t}(p_1q_1x^2 + p_2q_2y^2 - c')E'(t) + \lambda_1(t)\,[F(x) - q_1E'x^2] + \lambda_2(t)\,[G(y) - q_2E'y^2]\\
&= E'(t)[e^{-\delta t}(p_1q_1x^2 + p_2q_2y^2 - c') - \lambda_1q_1x^2 - \lambda_2q_2y^2] + \lambda_1(t)F(x) + \lambda_2(t)G(y) \qquad \text{.....(4.7)}
\end{aligned}$$

where $\lambda_1(t)$, $\lambda_2(t)$ are the adjoint variables to be determined suitably. Now the adjoint equation for $\lambda_1(t)$ is given by

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial x} = -e^{-\delta t}\cdot 2xp_1q_1E'(t) - \lambda_1(t)\,[r - \frac{2rx}{K} - q_1E'.2x] \qquad \text{.....(4.8)}$$

So, in the steady state, when $\dfrac{r\left(1-\frac{x}{K}\right)}{q_1x} = E' = \dfrac{s\left(1-\frac{y}{L}\right)}{q_2y}$, we have

$$\frac{d\lambda_1}{dt} = -e^{-\delta t}.2xp_1q_1E'(t) + r\lambda_1(t)$$

$$\text{or, } \frac{d\lambda_1}{dt} - r\lambda_1(t) = -2xp_1q_1e^{-\delta t}E'(t) \qquad .....(4.9)$$

From (4.9) it follows that $\lambda_1(t) = Ae^{rt} + \dfrac{2xp_1q_1e^{-\delta t}E'(t)}{\delta + r}$

Under the choice A = 0, it is proved that

$$e^{\delta t}\lambda_1(t) \to \frac{2xp_1q_1E'(t)}{\delta + r} \quad as\ t \to \infty \qquad .....\ (4.10)$$

Similarly, it is proved that,

$$e^{\delta t}\lambda_2(t) \to \frac{2yp_2q_2E'(t)}{\delta + s} \quad as\ t \to \infty \qquad .....(4.11)$$

Taking H to be maximum for some $E' \in V'_t$ (control set), we have $\dfrac{\partial H}{\partial E'} = 0$.

So, $(p_1q_1x^2 + p_2q_2y^2 - c') - e^{\delta t}(\lambda_1q_1x^2 + \lambda_2q_2y^2) = 0$

Hence the optimal path is given by,

$$p_1q_1x^2 + p_2q_2y^2 - c' - \left\{\frac{2p_1q_1^2x^3}{\delta + r} + \frac{2p_2q_2^2y^3}{\delta + s}\right\}E'_{\max} = 0 \qquad .....(4.12)$$

As $E'_{\max} = \dfrac{r\left(1-\frac{x}{K}\right)}{q_1x} = \dfrac{s\left(1-\frac{y}{L}\right)}{q_2y}$, so

$$\frac{E'_{\max}}{\delta + r} = \frac{r(K-x)}{Kq_1x(\delta + r)} \qquad .....(4.13)$$

$$\text{and } \frac{E'_{\max}}{\delta + s} = \frac{s(L-y)}{Lq_2y(\delta + s)} \qquad .....(4.14)$$

From (4.12), (4.13) and (4.14), it follows that the optimal path is finally given by (4.5). Next, the optimal biomasses x_δ and y_δ are determined by the interaction of (4.5) and (4.3).

Now from (4.3), $x = \frac{aKy}{(a-b)y+bL}$, putting $x = \frac{aKy}{(a-b)y+bL}$ in (4.5), we see that ultimately y_δ is determined by (4.6). Lastly, x_δ, $(E')^*$ and E^* are determined accordingly.

Theorem 4.3. Let the exploited system by given by (I). Let the total discounted revenue be given by II(B). Then the optimal path is given by

$$2p_1q_1Lx^3 + 2p_2q_2Ky^3 - KL(p_1q_1x^2 + p_2q_2y^2 - c') = 0 \qquad .....(4.16)$$

Further the optimal biomass y_δ is determined by the unique positive root of the equation

$$\begin{aligned}
&2p_2q_2K(a-b)^3y^6 + p_2q_2KL\,(7b-a)(a-b)^2y^5 + 3p_2q_2bKL^2(a-b)(3b-a)y^4\\
&+ [2p_2q_2Kb^3L^3 - 3p_2q_2KL^3(a-b)b^2 - c'KL(a-b)^3 - p_1q_1K^3La^2(a-b)\\
&+ 2p_1q_1La^3K^3]y^3 - [p_2q_2Kb^3L^4 + 3c'bKL^2(a-b)^2\\
&+ p_1q_1a^2bK^3L^2]y^2 - 3c'b^2(a-b)KL^3y - c'b^3KL^4 = 0 \qquad ..... (4.17)
\end{aligned}$$

and x_δ is given by the positive root of the equation

$$(a-b)xy_\delta + bLx - aKy_\delta = 0$$

The optimal effort $(E')^*$ is given by $y_\delta = \left(\frac{sL}{s+Lq_2(E')^*}\right)$ and E^* is determined (in principal value) from $(E')^*$.

Proof. The Hamiltonian for (I) and II(B) is given by

$$H = \frac{(p_1q_1x^2 + p_2q_2y^2 - c')E'(t)}{t^{\delta+2}} + \lambda_1(t)\,[F(x) - q_1E'x^2] + \lambda_2(t)[G(y) - q_2E'y^2]$$

$$= E'(t)[\frac{(p_1q_1x^2 + p_2q_2y^2 - c')}{t^{\delta+2}} - \lambda_1q_1x^2 - \lambda_2q_2y^2] + \lambda_1(t)\,F(x) + \lambda_2(t)G(y) \qquad ..... (4.18)$$

where $\lambda_1(t)$ and $\lambda_2(t)$ are the adjoint variables to be determined suitably. Now the adjoint equations for $\lambda_1(t)$ and $\lambda_2(t)$ are given by

$$\frac{d\lambda_1}{dt} - r\lambda_1(t) = -\frac{2xp_1q_1E'(t)}{t^{\delta+2}}$$

and $$\frac{d\lambda_2}{dt} - s\lambda_2(t) = -\frac{2yp_2q_2E'(t)}{t^{\delta+2}} \qquad ..... (4.19)$$

It is finally proved that

$$t^{\delta+2}\lambda_1(t) \rightarrow \frac{2xp_1q_1E'}{r} \quad \text{as } t \rightarrow \infty,$$

and

$$t^{\delta+2}\lambda_2(t) \rightarrow \frac{2yp_2q_2E'}{s} \quad \text{as } t \rightarrow \infty$$

Taking H to be maximum for some $E' \in V'_t$ (control set), we have $\frac{\partial H}{\partial E'} = 0$.

So, $(p_1q_1x^2 + p_2q_2y^2 - c') - t^{\delta+2}(\lambda_1q_1x^2 + \lambda_2q_2y^2) = 0$

Hence the optimal path is given by,

$$p_1q_1x^2 + p_2q_2y^2 - c' - \left\{\frac{2x^3p_1q_1^2}{r} + \frac{2y^3p_2q_2^2}{s}\right\} E'_{\max} = 0 \qquad \text{..... (4.21)}$$

As $E'_{\max} = \frac{r\left(1-\frac{x}{K}\right)}{q_1x} = \frac{s\left(1-\frac{y}{L}\right)}{q_2y}$.

So it follows from (4.21) that the equation of optimal path is

$2p_1q_1Lx^3 + 2p_2q_2Ky^3 - KL(p_1q_1x^2 + p_2q_2y^2 - c') = 0$.

Optimal biomasses x_δ and y_δ are determined by the intersection of (4.16) and (4.3). From (4.3), we get $x = \frac{aKy}{(a-b)y + bL}$, putting this value of x in (4.16) and simplifying, we finally get y_δ given by (4.17). x_δ, $(E')^*$ and E^* are determined accordingly.

5. SPECIAL CHOICE OF PARAMETERS.

$\alpha = 1,\ r = 3,\ q_1 = 2,\ p_1 = 3,\ K = 1$

$\beta = 2,\ s = 1,\ q_2 = 1,\ p_2 = 4,\ L = 2.$

Here $rLq_2 > sKq_1$, i.e., $a > b$. $c' < 22$. let $c' = 10$, $E'_{\max}$ is determined by $40E'^4 + 160E'^3 + 150E'^2 + 34E' - 19 = 0$. It is approximately equal to 0.559.

Case II(A).

Let $\delta = 2$.

Then from (4.5) it follows that the optimal path is given by,

$40y^6 + 160y^5 + 240y^4 + 508y^3 - 941y^2 - 900y - 300 = 0$.

Also (x_δ, y_δ) is approximately equal to (0.85, 1.3), $(E')^*$ is approximately equal to = 0.269. Hence $(E')^* < E'_{max}$.

So $(E')^* \in V'_t = [E'_0, E'_{max}]$ and hence $E^* \in V_t = [E_{t_0}, E'_{max}]$.

This verifies our results for the case a > b.

Again, parameters may be so choosen that a < b holds and in this case also, the results may be similarly verified.

Case II(B).

From (4.16), it follows that the optimal path is given by $8y^6 + 16y^5 + 18y^3 - 95y^2 - 60y - 20 = 0$. Also (x_δ, y_δ) is approximately equal to (0.9, 1.5). $(E')^*$ is approximately equal to 0.167. Hence $(E')^* < E'_{max}$. As t_0 may be any suitable positive real, so $E'(t_0)$ may be chosen accordingly and $(E')^* \in V'_t = [E_{t_0}, E'_{max}]$ and hence $E^* \in V_t = [E_{t_0}, E'_{max}]$. This verifies our results for the case a > b.

Again parameters may be so chosen that a < b holds and in this case also, the results may be similarly verified.

6. DISCUSSIONS

(i) It follows from (4.1), that the upper threshold value of c′ and hence that of c depends on the given parameters only. Further, at all points on L′, where c′ is not kept under this upper limit, effort becomes negative; so in this case, exploitation becomes impossible. From (4.2), it follows that E'_{max} is different for different choices of c′, and as such, it also depends on the given parameters only. Further, unless $E_{max}/\mathrm{Sup}(E_{max})$ is determined, the control set V_t remains unknown and so optimization loses its meaning.

(ii) Apart from economic justifications, the depreciation $e^{-\delta t}$ is introduced, in calculating the discoundted revenue J, in order to assure the convergence of the integral of J. Naturally, the case $\delta = 0$ remains indecisive, as the corresponding integral may be divergent. Again, the auxiliary functions $\lambda_i(t)$, i = 1, 2, of the Hamiltonian for J are found to satisfy the transversality condition $e^{-\delta t}\lambda_i(t) \to$ finite quality as $t \to \infty$. This condition does not appear to be justified mathematically [1].

However, if $\frac{1}{t^{\delta+2}}$ is taken as the depreciation when the integral of J is $0(t^\delta)$, $\delta > 0$, then from (4.10) and (4.11), it folows that $\lambda_i(t)$ satisfy weaker transversality conditions.

Further, it is always justified to take even $\delta = 0$. From these considerations, it is preferable to take $\frac{1}{t^{\delta+2}}$ in place of $e^{-\delta t}$.

Lastly from (4.5) it follows that the optimal path depends on the choice of δ, whereas from (4.16) it is seen that the optimal path is independent of δ. So the optimal effort E^* depends on δ in case II(A), but it is independent of δ in case II(B). Naturally, in the latter case, there should not be any difficulty in verifying that $E^* \in V_t$ (which is independent of δ). In fact, it is verified in p. 11 for the case II(B). On the other hand, in the former case, the question of verifying whether $E^* \in V_t$ or not, does not arise when δ is arbitrary. Moreover, if for some δ, $E^* \notin V_t$, then also there may be some δ (which may be found by chance) for which $E^* \in V_t$. In fact, it is shown that $E^* \notin V_t$ when $\delta = 2$, but $E^* \notin V_t$ when $\delta = 1$. This is the major difficulty with $e^{-\delta t}$ where δ is not specific. Again such specific δ can not be assertained at the very outset. This is why, it is better to take the discount $1/t^{\delta+2}$ which is self adjusting in nature and which varies from model to model.

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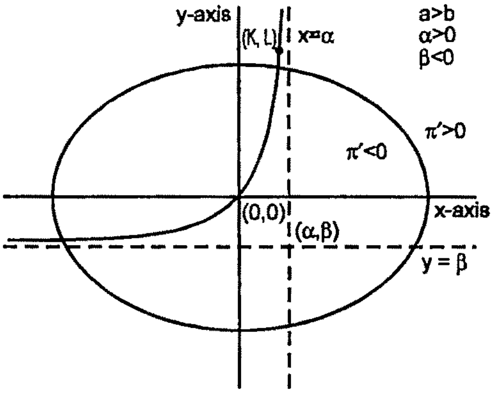
y-axis
(K, L)
x=α
a>b
α>0
β<0
π'>0
π'<0
(0,0)
(α,β)
x-axis
y = β

Fig. 1

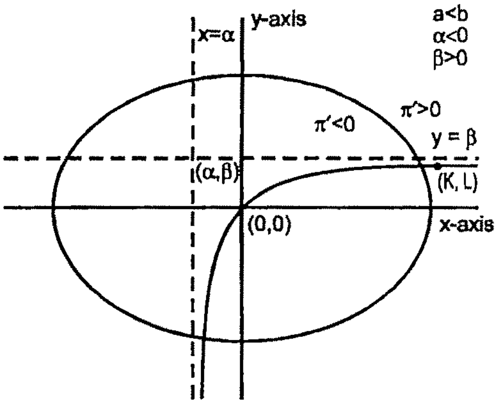
x=α
y-axis
a<b
α<0
β>0
π'>0
π'<0
y = β
(α,β)
(K, L)
(0,0)
x-axis

Fig. 2

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1. N. L. Alling, Foundations of Analysis on Surreal Number Fields, North-Holland Publishing Co., 1987.
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